Lecture 01
01/18/12

Introduction (1.1):

$$\begin{tikzcd} {[\underline{x}(0)]_{\mathcal{B}}} \arrow[r] & {[\underline{x}(t)]_{\mathcal{B}}} \arrow[d] \\ \underline{x}(0) \arrow[u] \arrow[r, dashed, "\text{Dynamics.}"] & \underline{x}(t) \end{tikzcd}$$

$\mathcal{B}$ Basis

Ex

$$\begin{cases} x_1(n) = 2x_1(n-1) + x_2(n-1) \\ x_2(n) = x_1(n-1) + 2x_2(n-1) \end{cases} \Rightarrow \begin{array}{l} x_1(n) = ? \\ x_2(n) = ? \end{array}$$

$x_{1,2}(0)$ given.

Change of variables: $y_1 = \dfrac{x_1 + x_2}{2}$, $y_2 = \dfrac{x_1 - x_2}{2}$

$$\Rightarrow \begin{cases} y_1(n) = 3y_1(n-1) \\ y_2(n) = y_2(n-1) \end{cases} \Rightarrow \begin{array}{l} y_1(n) = 3^n y_1(0) \\ y_2(n) = y_2(0) \end{array}$$

Converting back to x_1, x_2 gives

$$\left| \begin{array}{l} x_1(n) = \frac{1}{2}(3^n + 1)\, x_1(0) + \frac{1}{2}(3^n - 1)\, x_2(0) \\ x_2(n) = \frac{1}{2}(3^n - 1)\, x_1(0) + \frac{1}{2}(3^n + 1)\, x_2(0) \end{array} \right|.$$

To recap: $\underline{x} = (x_1, x_2)^T$, $\underline{y} = (y_1, y_2)^T$

$$\underline{x}(n) = \underset{\substack{\parallel \\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}}{A}\, \underline{x}(n-1) \xrightarrow[\text{(decoupling)}]{\text{diagonalization}} \underline{y}(n) = \underset{\substack{\parallel \\ \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}}{D}\, \underline{y}(n-1).$$

Vector spaces (2.1):

$\underline{x}$ vector, c scalar.

Def. V is a vector space if it is closed under the operations of addition and scalar multiplication (i.e., for any $\underline{x}, \underline{y} \in V$ and scalar c, $\underline{x}+\underline{y} \in V$ and $c\underline{x} \in V$), and satisfies the axioms on [p. 10–11, Sadun].

V can either be finite-dim. or infinite-dim.

$\longrightarrow$ depends on # of degrees of freedom.

V called
- real vector space if $c \in \mathbb{R}$
- complex vector space if $c \in \mathbb{C}$.

Ex. $$V = \Big\{ \underline{x} : \underbrace{\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{\left(\text{i.e., } \underline{x} = (x_1, \dots, x_n)^T\right)} \Big\} \doteq \mathbb{R}^n$$

with $$\underline{x} + \underline{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \quad c\underline{x} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}$$

where $x_i, c \in \mathbb{R}$.

Ex. Same as above, but with $x_i, c \in \mathbb{C}$

$\Rightarrow V \doteq \mathbb{C}^n$.

Ex.
$$M_{nm} \doteq \left\{ \underline{a} : \underline{a} = \begin{array}{cc} \left(\begin{array}{ccc} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{array} \right) & \updownarrow n \\ \xleftrightarrow{\quad m \quad} & \end{array} \right\}$$

with
$$\underline{a} + \underline{b} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{pmatrix}$$
$$c\underline{a} = \begin{pmatrix} ca_{11} & \cdots & ca_{1m} \\ \vdots & \ddots & \vdots \\ ca_{n1} & \cdots & ca_{nm} \end{pmatrix}.$$

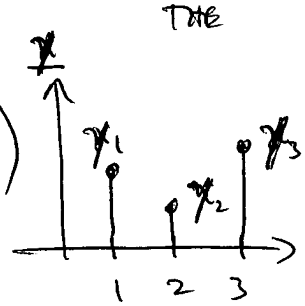
where $a_{ij}, c \in \mathbb{R}$.

Ex.
$$C^0[0,1] \doteq \{ f : f : [0,1] \to \mathbb{R} \text{ is continuous} \}$$

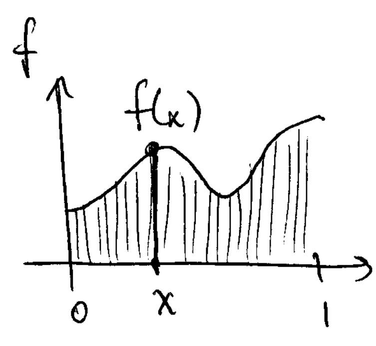
with
$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x).$$

<u>NOTE</u>: $C^0[0,1]$ is an <u>infinite-dim.</u> vector space since it takes an infinite # of pieces of information to specify an element of the vector space.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$



vs.



$$\underline{x} = \underline{x}(x),\ x \in \underbrace{\{1,2,3\}}_{\text{finite set}}.$$

$$f = f(x),\ x \in \underbrace{[0,1]}_{\text{infinite set.}}$$

EX. $\mathbb{R}_n[t] \doteq \{ p : \underbrace{p(x) = a_0 + a_1 x + \dots + a_n x^n}_{p \text{ POLYNOMIAL OF DEGREE } \leq n} \}$

WITH $(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + \dots + (a_n+b_n)x^n$.

AND $(cp)(x) = cp(x)$,

WHERE $a_i, c \in \mathbb{R}$.

NOTE: $\mathbb{R}_n[t]$ IS FINITE-DIM. SINCE WE ONLY NEED TO SPECIFY $(n+1)$ COEFFICIENTS $\{a_i\}_{i=0}^n$ TO DESCRIBE ANY ELEMENT.

CONTRAST THIS TO:

EX. $\mathbb{R}[t] \doteq \{ p : p$ IS A POLYNOMIAL OF ANY DEGREE$\}$.

$\mathbb{R}[t]$ IS INF.-DIM. SINCE TO SPECIFY AN ELEMENT ONE NEEDS TO KNOW AN INFINITE # OF COEFF. $\{a_i\}_{i=0}^{\infty}$ (EVEN IF ALL BUT A FINITE NO. OF THEM ARE ZERO!).

LECTURE 02
01/20/12

SUBSPACES:

DEF. $U \subset V$ IS A SUBSPACE OF V IF IT IS CLOSED UNDER ADDITION AND SCALAR MULTIPLICATION INHERITED FROM V.

TRIVIAL SUBSPACES: $\{\underline{0}\}$, V.

EX. $V = \mathbb{R}^2$

- $U_1 = \{\underline{x} \in \mathbb{R}^2 : x_1 + x_2 = 0\}$ SUBSPACE SINCE
$$\underline{x}+\underline{y} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} \in U_1 \quad \left(\begin{array}{l}\text{BECAUSE} \\ (x_1+y_1)+(x_2+y_2) = 0\end{array}\right)$$
$$c\underline{x} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} \in U_1 \quad \left(\begin{array}{l}\text{BECAUSE} \\ cx_1 + cx_2 = 0\end{array}\right)$$

- $U_2 = \{\underline{x} \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ NOT A SUBSPACE SINCE $\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $y \in \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ARE BOTH IN U_2, BUT $\underline{x}+\underline{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin U_2$ SINCE $1+1 \neq 1$.

- $\mathbb{Z}^2 \doteq \{\underline{x} \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{Z}\}$ NOT A SUBSPACE OF $\mathbb{R}^2$ SINCE $c\underline{x} \notin \mathbb{Z}^2$ IF $c \notin \mathbb{Z}$.

(NOTE THAT WHILE $\mathbb{Z}^2$ IS NOT A SUBSPACE OF $\mathbb{R}^2$, $\mathbb{Z}^2$ IS A VECTOR SPACE IF WE DEFINE SCALAR MULTIPLICATION WITH $c \in \mathbb{Z}$, NOT $c \in \mathbb{R}$.)

EX. $\mathbb{R}[t]$ IS A SUBSPACE OF $C^0[0,1]$.

SPACE OF ALL REAL-VALUED POLYNOMIALS

$\mathbb{R}_n[t]$ IS A SUBSPACE OF $\mathbb{R}[t]$ (AND OF $C^0[0,1]$).

SPACE OF ALL REAL-VALUED POLYNOMIALS W/ DEGREE $\leq n$.

BASIS AND DIMENSION (2.2):

$\mathcal{B} \doteq \{\underline{b}_1, \ldots, \underline{b}_n\}$ ORDERED SET OF VECTORS IN V.

- WE SAY $\underline{v}$ IS A LINEAR COMBINATION OF $\mathcal{B}$ IF $\underline{v} = a_1\underline{b}_1 + \cdots + a_n\underline{b}_n$ FOR SOME SCALARS $\{a_i\}_{i=1}^n$.
- $\mathcal{B}$ LINEARLY INDEPENDENT IF $a_1\underline{b}_1 + \cdots + a_n\underline{b}_n = 0 \iff a_i = 0$ FOR ALL $i=1,\ldots,n$.

THM. IF $n>1$, $\mathcal{B}$ LINEARLY INDEP. $\iff$ b_i ~~IS~~ A LINEAR COMBINATION OF $\{b_j\}_{j\neq i}$ FOR SOME $i \in \{1,\ldots,n\}$.
(I.E., AN ELEMENT OF $\mathcal{B}$ CAN BE WRITTEN IN TERMS OF THE OTHERS.)

PF. EASY.

NOTE: $\mathcal{B}$ LINEARLY DEP. IMPLIES ONE $\underline{b}_i$ IS LINEAR COMB. OF OTHERS, BUT DOES NOT SAY WHICH ONE.

EX.

$$\overbrace{\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{INDEP.}}, \begin{pmatrix}0\\1\\1\end{pmatrix}}^{\text{DEPENDENT.}}$$

DEPENDENT

INDEP.

- $\text{SPAN}(\mathcal{B}) \doteq \{\underline{v} \in V : \underline{v}$ LINEAR COMBINATION OF $\mathcal{B}\}$

THM. $\mathcal{B} = \{\underline{b}_1, .., \underline{b}_n\}$ FINITE. THEN, $\text{SPAN}(\mathcal{B}) \subset V$ IS A SUBSPACE OF V.

PF. EASY.

NOTE: $\text{SPAN}(\mathcal{B})$ IS THE SMALLEST SUBSPACE THAT CONTAINS $\mathcal{B}$. THAT IS,

$$\text{SPAN}(\mathcal{B}) = \bigcap_{\substack{\{\lambda : U_\lambda \text{ SUBSPACE} \\ \text{AND } \mathcal{B} \subset U_\lambda\}}} U_\lambda .$$

DEF. IF $\mathcal{B}$ SUCH THAT

(i) $\text{SPAN}(\mathcal{B}) = V$

(ii) $\mathcal{B}$ LINEARLY INDEP.

WE SAY $\mathcal{B}$ IS A BASIS OF V.

EX. $V = \mathbb{R}^n$

- $\mathcal{E} = \{\underline{e}_1, \dots, \underline{e}_n\}$, $\underline{e}_j = (0, \dots, 0, \underset{j^{th}\text{ ENTRY}}{1}, 0, \dots, 0)^T$

  IS THE STANDARD BASIS OF $\mathbb{R}^n$.

$V = \mathbb{C}^n$

- $\mathcal{E}$ IS BASIS IF V IS CONSIDERED A COMPLEX VECTOR SPACE (I.E, SCALAR MULTIPLICATION BY $c \in \mathbb{C}$).

- $\mathcal{E}$ NOT A BASIS IF V IS CONSIDERED A REAL VECTOR SPACE (I.E, SCALAR MULTIPLICATION BY $c \in \mathbb{R}$).

  IN THIS CASE,

  $$\tilde{\mathcal{E}} = \{\underline{e}_1, \dots, \underline{e}_n, \underline{f}_1, \dots, \underline{f}_n\}$$

  WITH $\underline{e}_j$ AS BEFORE AND

  $\underline{f}_j = (0, \dots, 0, \underset{j^{th}\text{ ENTRY}}{i}, 0, \dots, 0)^T$ IS A BASIS.

  $i = \sqrt{-1}$

EX. $V = \mathbb{R}_n[t]$

- $\mathcal{B} = \{1, t, t^2, \dots, t^n\}$ IS A BASIS.
- $\mathcal{B}_1 = \{2, 3t - t^2, 4t^2, t^3, \dots, t^n\}$ IS A BASIS.
- $\mathcal{B}_2 = \{1, \underbrace{t+t^2, 2t+2t^2}, t^3, \dots, t^n\}$ IS <u>NOT</u> A BASIS.

(SPAN$(\mathcal{B}_2) \neq V$ AND $\mathcal{B}_2$ LINEARLY DEP.)

$\tilde{\mathcal{B}}_2 = \{1, t, \underbrace{t+t^2, 2t+t^2}, t^3, \dots, t^n\}$ IS <u>NOT</u> A BASIS

($\tilde{\mathcal{B}}_2$ LINEARLY DEP.)

EX. $V = M_{22}$

- $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

IS STANDARD <u>BASIS</u> OF M_{22}.

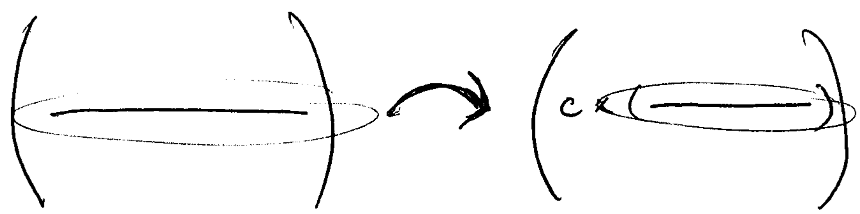
<u>REMARK</u>:

$$\underbrace{a_1 \underline{b}_1 + \dots + a_n \underline{b}_n}_{\substack{\text{LINEAR COMBINATION} \\ \text{OF } \{\underline{b}_1, \dots, \underline{b}_n\} \subset \mathbb{R}^n}} = \underbrace{\begin{pmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_n \\ | & & | \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{\underline{x}}$$

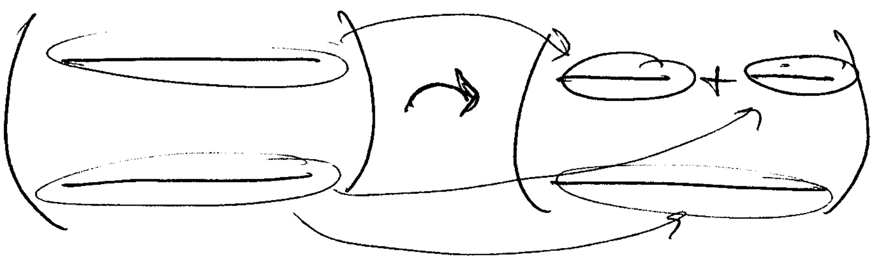
(A = MATRIX W/ COLUMNS $\underline{b}_1, \dots, \underline{b}_n$)

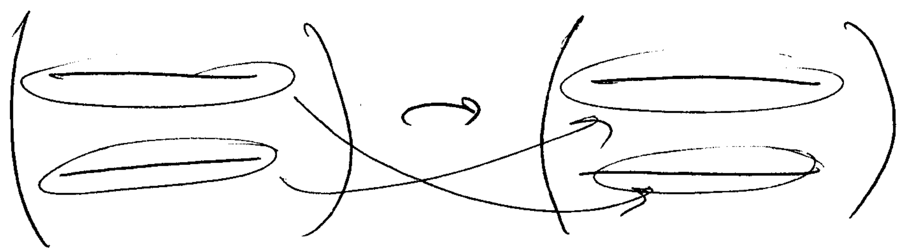
SO, SPAN$(\mathcal{B})$ = COLUMN SPACE OF A.

FOR ANY MATRIX A, DEFINE ROW OPERATIONS:

(i)  FOR SCALAR c.

[MULTIPLY ROW]

(ii)  [ADD ROWS]

(iii)  [SWITCH ROWS]