Lecture 28
04/02/12

Suppose V is an inner product space and $L : V \to V$ is a linear operator which is diagonalizable.

$\Rightarrow$ V has basis $\mathcal{B}$ of eigenvectors of L.

Q: Is $\mathcal{B}$ an orthogonal basis?

A: Generally not.

Ex. $V = \mathbb{R}^2$ w/ std. inner product.

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in M_{2,2}(\mathbb{R}) \quad (\text{i.e., } A : V \to V).$$

E-values of A are $\lambda = 0$ and 3, with e-vectors $\underline{\xi}_1 = (2,-1)^T$ and $\underline{\xi}_2 = (1,1)^T$.

However, $\langle \underline{\xi}_1 | \underline{\xi}_2 \rangle = (2,-1)(1,1)^T = 1 \neq 0$,

so $\underline{\xi}_1 \not\perp \underline{\xi}_2$.

Q: Given an inner product space, which class of operators generate an <u>orthogonal</u> basis of e-vectors?

What is geometric structure of these operators?

ADJOINTS (7.1):

<u>DEF.</u> V INNER PROD. SPACE, $L : V \to V$.

THE <u>ADJOINT</u> OPERATOR L^* OF L IS THE UNIQUE OPERATOR THAT SATISFIES

$$\langle L^* \underline{x} \mid \underline{y} \rangle = \langle \underline{x} \mid L \underline{y} \rangle \quad \text{FOR } \underline{\text{ALL}} \ \underline{x}, \underline{y} \in V.$$

- IF $V = \mathbb{C}^n$ w/ STD. INNER PRODUCT, THE ADJOINT OF $A \in M_{n,n}(\mathbb{C})$ IS $A^* = \overline{A}^T$.
- FOR A GENERAL VECTOR SPACE w/ <u>ORTHONORMAL</u> BASIS $\mathcal{E} = \{\underline{e}_i\}_{i=1}^n$:

MATRIX REPRESENTATION OF $L : V \to V$ IS

$$([L]_{\mathcal{E}})_{ij} = \langle \underline{e}_i \mid L \underline{e}_j \rangle$$

(EASY TO CHECK THIS, AND THAT $L = \sum_{i=1}^n \sum_{j=1}^n ([L]_{\mathcal{E}})_{ij} \, |\underline{e}_i\rangle\langle \underline{e}_j|$).

THEN,

$$\begin{aligned}
([L^*]_{\mathcal{E}})_{ij} = \langle \underline{e}_i \mid L^* \underline{e}_j \rangle &= \overline{\langle L^* \underline{e}_j \mid \underline{e}_i \rangle} \\
&= \overline{\langle \underline{e}_j \mid L \underline{e}_i \rangle} \\
&= \overline{([L]_{\mathcal{E}})_{ji}}
\end{aligned}$$

$$\Rightarrow \quad [L^*]_{\mathcal{E}} = [L]_{\mathcal{E}}^{*} .$$

EX. $V = \mathbb{C}^3$, w/ STD. INNER PROD.

LET $L\underline{x} = (3x_1 + ix_2,\ ix_2 - 2x_3,\ (1+i)x_1 + 5x_3)^T$

WHERE $\underline{x} = (x_1, x_2, x_3)^T \in \mathbb{C}^3$.

Q: WHAT IS $L^*\underline{x}$?

A: $$L = \begin{pmatrix} 3 & i & 0 \\ 0 & i & -2 \\ 1+i & 0 & 5 \end{pmatrix} \Rightarrow L^* = \bar{L}^T = \begin{pmatrix} 3 & 0 & 1-i \\ -i & -i & 0 \\ 0 & -2 & 5 \end{pmatrix}$$

$$\Rightarrow L^*\underline{x} = (3x_1 + (1-i)x_3,\ -ix_1 - ix_2,\ -2x_2 + 5x_3)^T.$$

EX. $V = L^2(\mathbb{R})$ (INFINITE DIMENSIONAL SPACE OF SQUARE INTEGRABLE FCN'S)

I.E, $f \in L^2(\mathbb{R}) \Leftrightarrow \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$.

LET $\langle f | g \rangle = \int_{-\infty}^{\infty} \overline{f(x)}\, g(x)\, dx$ BE THE INNER PRODUCT (THIS IS WELL DEFINED SINCE

$$|\langle f|g\rangle| \leq \|f\|\,\|g\| = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)^{1/2} < \infty .\ )$$

IT IS EASY TO CHECK THAT $L = \frac{d}{dx}$ IS A LINEAR OPERATOR ON V (ASSUMING THAT ~~ALL~~ FCN'S IN V ARE DIFFERENTIABLE).

Q: WHAT IS L^*?

A: $$\langle f | L g \rangle = \int_{-\infty}^{\infty} f(x) \left(\frac{d}{dx} g(x) \right) dx$$

INTEGRATION BY PARTS

$$= \underbrace{\Big[f(x) g(x) \Big]_{-\infty}^{\infty}}_{\substack{=0 \text{ SINCE SQUARE INTEGRABILITY} \\ \Rightarrow f(x),\, g(x) \to 0 \\ \text{AS } x \to \pm\infty}} - \int_{-\infty}^{\infty} \left(\frac{d}{dx} f(x) \right) g(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left(-\frac{d}{dx} f(x) \right) g(x)\, dx$$

$$= \langle L^* f | g \rangle$$

SO, $L^* = -\frac{d}{dx}$.

- SIMILARLY, IF $\tilde{L}$ WERE $\frac{d^2}{dx^2}$, WE WOULD FIND THAT $\tilde{L}^* = \frac{d^2}{dx^2}$ BY INTEGRATING BY PARTS TWICE. IN THIS CASE, $\tilde{L}^* = \tilde{L}$ AND $\tilde{L}$ WOULD BE CALLED <u>SELF ADJOINT</u>.

PROPERTIES OF ADJOINTS :-

$$\text{(i)} \quad (A+B)^* = A^* + B^*$$

$$\text{(ii)} \quad (AB)^* = B^* A^*$$

$$\text{(iii)} \quad (cA)^* = \bar{c} A^*$$

$$\text{(iv)} \quad (A^*)^* = A.$$

<u>THM.</u>

$$\text{(i)} \quad (\operatorname{Ker} A^*) = (\operatorname{Ran} A)^\perp$$

$$\text{(ii)} \quad (\operatorname{Ker} A) = (\operatorname{Ran}(A^*))^\perp$$

$$\text{(iii)} \quad (\operatorname{Ker} A^*)^\perp = \operatorname{Ran} A$$

$$\text{(iv)} \quad (\operatorname{Ker} A)^\perp = \operatorname{Ran} A^*.$$

<u>PF.</u> (i)

$$\underline{x} \in (\operatorname{Ran} A)^\perp \iff \langle \underline{x} \mid A \underline{y} \rangle = 0 \text{ for all } \underline{y}$$

$$\iff \langle A^* \underline{x} \mid \underline{y} \rangle = 0 \text{ for all } \underline{y}$$

$$\iff A^* \underline{x} = 0$$

$$\iff \underline{x} \in \operatorname{Ker} A^*.$$

Then, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (iii) and we are done.

Lecture 29
04/04/12

Def. An operator $L: V \to V$ on an inner product space V is <u>self-adjoint</u> (or <u>Hermitian</u>) if $L^* = L$.

- If V is a <u>real</u> inner product space, we have that $L^T = L$ and L is called <u>symmetric</u>.

Thm. Every eigenvalue of a self-adjoint operator is real.

Pf. Let λ be an e-value of L with e-vector $\underline{v}$. Then,

$$\lambda \|\underline{v}\|^2 = \langle \underline{v} | \lambda \underline{v} \rangle = \langle \underline{v} | L\underline{v} \rangle = \langle L^* \underline{v} | \underline{v} \rangle$$
$$= \langle L\underline{v} | \underline{v} \rangle = \langle \lambda \underline{v} | \underline{v} \rangle = \bar{\lambda} \|\underline{v}\|^2.$$

$$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}.$$

Thm. Eigenspaces of distinct eigenvalues are orthogonal.

Pf. Let λ, μ be distinct e-values with corresponding eigenspaces E_λ and E_μ. Let $\underline{v} \in E_\lambda$, $\underline{w} \in E_\mu$. Then,

$$\langle L\underline{v} | \underline{w} \rangle = \bar{\lambda} \langle \underline{v} | \underline{w} \rangle = \lambda \langle \underline{v} | \underline{w} \rangle$$

since $\lambda \in \mathbb{R}$.

$$\|$$

$$\langle \underline{v} | L\underline{w} \rangle = \mu \langle \underline{v} | \underline{w} \rangle$$

so $\underbrace{(\lambda - \mu)}_{\neq 0} \langle \underline{v} | \underline{w} \rangle = 0$, and $\langle \underline{v} | \underline{w} \rangle = 0 \Rightarrow \underline{v} \perp \underline{w}$.

NOTE: EIGENVECTORS CORRESPONDING TO SAME EIGENVALUE DON'T HAVE TO BE ORTHOGONAL!

EX.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \lambda = 1, -1$$

$$\Rightarrow \underbrace{\underline{\xi}_1 = (1,0,0)^T, \ \underline{\xi}_2 = (1,1,1)^T}_{\text{CORRESPONDING TO } \lambda = 1}, \ \underbrace{\underline{\xi}_3 = (0,1,-1)^T}_{\text{CORRESPONDING TO } \lambda = -1.}$$

EASY TO SEE $\underline{\xi}_3 \perp \operatorname{SPAN}\{\underline{\xi}_1, \underline{\xi}_2\}$. ✓

BUT $\underline{\xi}_1 \not\perp \underline{\xi}_2$! HOWEVER, $\underline{\xi}_1$ AND $\underline{\xi}_2$ FORM A <u>BASIS</u> OF E_1, SO WE CAN ORTHOGONALIZE BY GRAM-SCHMIDT.

$$\Rightarrow \underline{b}_1 = \underline{\xi}_1 = (1,0,0)^T$$

$$\underline{b}_2 = (I - P_{\underline{b}_1})\underline{\xi}_2 = \underline{\xi}_2 - P_{\underline{b}_1}\underline{\xi}_2$$

$$= \underline{\xi}_2 - \frac{\langle \underline{b}_1 | \underline{\xi}_2 \rangle}{\|\underline{b}_1\|^2}\underline{b}_1$$

$$= (1,1,1)^T - \frac{1}{1}(1,0,0)^T$$

$$= (0,1,1)^T$$

⟵ STILL AN EIGENVECTOR CORRESPONDING TO $\lambda = 1$!

$$\left.\begin{aligned} \Rightarrow \underline{e}_1 &= \frac{\underline{b}_1}{\|\underline{b}_1\|} = (1,0,0)^T \\ \underline{e}_2 &= \frac{\underline{b}_2}{\|\underline{b}_2\|} = \frac{1}{\sqrt{2}}(0,1,1)^T \\ \underline{e}_3 &= \frac{\underline{\xi}_3}{\|\underline{\xi}_3\|} = \frac{1}{\sqrt{2}}(0,1,-1)^T \end{aligned}\right\} \text{ORTHONORMAL BASIS OF EIGENVECTORS OF } A.$$

THM. LET L BE SELF-ADJOINT. ASSUME L IS DIAGONALIZABLE. THEN THERE IS AN ORTHONORMAL BASIS OF V CONSISTING OF EIGENVECTORS OF L.

PF. L HAS DISTINCT E-VALUES $\lambda_1, \ldots, \lambda_r$, $r \leq n$, WITH EIGENSPACES $E_1, \ldots, E_r$. THEN ALL THE E-SPACES ARE ORTHOGONAL. NOW USE GRAM-SCHMIDT TO FIND ORTHONORMAL BASIS $\mathcal{E}_i$ FOR EACH E_i. SINCE L IS DIAGONALIZABLE, $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_r$ IS A BASIS OF ORTHONORMAL E-VECTORS FOR V.

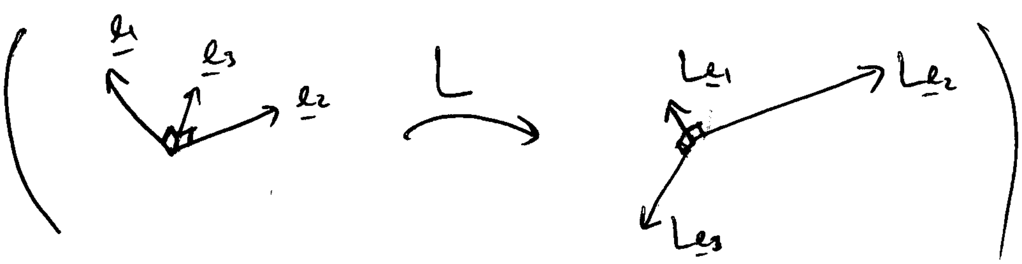
NOTE: WE WILL SHOW LATER THAT L SELF-ADJOINT IMPLIES L IS DIAGONALIZABLE, SO THE ASSUMPTION ABOVE CAN BE DROPPED.

THEREFORE,

THM. L SELF-ADJOINT $\iff$ $L = UDU^{-1}$

COLUMNS ARE ORTHONORMAL EVECTORS OF L.

DIAGONAL MATRIX OF REAL EIGENVALUES OF L



PF. "$\Rightarrow$" ALREADY DONE.

"$\Leftarrow$" $L = UDU^{-1}$ WITH D REAL DIAGONAL

$$U = \underbrace{(e_1, \dots, e_n)}_{\text{ORTHONORMAL BASIS } \mathcal{E}}.$$

SO, $[L]_{\mathcal{E}} = D = D^* = [L]_{\mathcal{E}}^* = [L^*]_{\mathcal{E}}$,

AND $L^* = L$.

COROLLARY: (REAL SPECTRAL THM.)

IF V REAL INNER PRODUCT SPACE,

$$S \text{ SYMMETRIC} \iff S = ODO^{-1}$$

ORTHONORMAL E-VECTORS OF S.

DIAGONAL MATRIX OF REAL EIGENVALUES

Q: IS EVERY OPERATOR WHICH HAS AN ORTHONORMAL BASIS OF E-VECTORS SELF-ADJOINT?

A: NO. FOR EXAMPLE, ANY $L = UDU^{-1}$.

ORTHONORMAL E-VECTORS

DIAGONAL MATRIX W/ COMPLEX ENTRIES.

EX $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$\Rightarrow \lambda_1 = 1 + 2i, \ \lambda_2 = 1 - 2i$ A NOT SELF-ADJOINT SINCE $\lambda \notin \mathbb{R}$

$\underline{e}_1 = \frac{1}{\sqrt{2}}(1, i)^T, \ \underline{e}_2 = \frac{1}{\sqrt{2}}(1, -i)^T$

$$\Rightarrow \langle \underline{e}_1 | \underline{e}_2 \rangle = 0 \Rightarrow \mathbf{A} = \underbrace{(\underline{e}_1 \ \underline{e}_2)}_{U} \overbrace{\begin{pmatrix} 1+2i & \\ & 1-2i \end{pmatrix}}^{D} \underbrace{(\underline{e}_1 \ \underline{e}_2)^{-1}}_{U^{-1}}$$

Lecture 30
04/06/12

Last time, we saw that

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$ is <u>not</u> self-adjoint, but still has an orthonormal basis of eigenvectors.

<u>Note:</u>
$$A^* A = \begin{pmatrix} 1 & +2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ +2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$
$$\parallel$$
$$A A^* = \begin{pmatrix} 1 & -2 \\ +2 & 1 \end{pmatrix}\begin{pmatrix} 1 & +2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

<u>Normal operators.</u>

<u>Def.</u> $N : V \to V$ is <u>normal</u> if $N^* N = N N^*$ (i.e., N commutes with its adjoint).

<u>Note:</u> L self-adjoint $\Rightarrow$ L normal

<u>Pf:</u> $L = L^* \Rightarrow L^* L = L L^*$.

<u>Properties:</u>

- N normal $\Leftrightarrow$ $\| N \underline{x} \| = \| N^* \underline{x} \|$ for all $\underline{x} \in V$.

<u>Pf.</u> N normal $\Leftrightarrow$ $N^* N - N N^* = 0$

$$\Leftrightarrow \langle (N^*N - NN^*)\underline{x} \mid \underline{x} \rangle = 0 \quad \text{FOR ALL } \underline{x} \in V$$

(USING THAT $N^*N - NN^*$ IS SELF-ADJOINT AND THAT FOR ANY SELF-ADJOINT L, $\langle L\underline{x} \mid \underline{x}\rangle = 0$ FOR ALL $\underline{x} \in V \Leftrightarrow L = 0$).

$$\Leftrightarrow \langle N\underline{x} \mid N\underline{x} \rangle = \langle N^*\underline{x} \mid N^*\underline{x} \rangle$$

$$\Leftrightarrow \|N\underline{x}\| = \|N^*\underline{x}\|.$$

- SUPPOSE N NORMAL. THEN λ EIGENVALUE OF N WITH E-VECTOR $\underline{v}$ $\Leftrightarrow$ $\bar{\lambda}$ EIGENVALUE OF N^* WITH E-VECTOR $\underline{v}$.

PF.

$$0 = \|\underbrace{(N - \lambda I)}_{\text{NORMAL}}\underline{v}\| = \|(N-\lambda I)^* \underline{v}\|$$
$$= \|(N^* - \bar{\lambda} I)\underline{v}\|$$

$$\left(\text{SINCE } \begin{aligned} (N-\lambda I)^*(N-\lambda I) &= N^*N - \bar{\lambda}N - \lambda N^* + \lambda\bar{\lambda} I \\ &= NN^* - \bar{\lambda}N - \lambda N^* + \lambda\bar{\lambda} I \\ &= (N-\lambda I)(N-\lambda I)^* \end{aligned} \right).$$

- N NORMAL $\Rightarrow$ EIGENSPACES OF DISTINCT EIGENVALUES ARE ORTHOGONAL.

PF. SUPPOSE λ, μ DISTINCT E-VALUES WITH CORRESPONDING EIGENSPACES E_λ AND E_μ. LET $\underline{v} \in E_\lambda$ AND $\underline{w} \in E_\mu$.

$$\left.\begin{aligned} N\underline{v} &= \lambda \underline{v} \\ N\underline{w} &= \mu \underline{w} \end{aligned}\right\} \Rightarrow (\lambda \overset{\neq 0}{-} \mu)\langle \underline{v} | \underline{w} \rangle$$

$$= \langle \bar{\lambda}\underline{v} | \underline{w} \rangle - \langle \underline{v} | \mu \underline{w} \rangle$$

$$= \langle N^* \underline{v} | \underline{w} \rangle - \langle \underline{v} | N \underline{w} \rangle$$

$$= \langle \underline{v} | N\underline{w} \rangle - \langle \underline{v} | N \underline{w} \rangle$$

$$= 0.$$

$$\Rightarrow \langle \underline{v} | \underline{w} \rangle = 0 \Rightarrow \underline{v} \perp \underline{w}.$$

BEFORE WE PROVE OUR MAIN RESULT, LET US INTRODUCE THE <u>SCHUR DECOMPOSITION</u> OF A MATRIX.

<u>THM.</u> (SCHUR DECOMPOSITION) ANY $A \in M_{n,n}(\mathbb{C})$ CAN BE WRITTEN AS $A = UTU^{-1}$.

ORTHONORMAL COLUMNS. — UPPER TRIANGULAR MATRIX

<u>PF.</u> (BY INDUCTION)

<u>CASE $n=1$</u>: $A \in \mathbb{C}$ ✓

<u>CASE $n-1$</u>: ASSUME TRUE.

<u>CASE n</u>: LET λ_1 BE E-VALUE OF A, W/ E-VECTOR $\underline{b}_1$, $\|\underline{b}_1\| = 1$. NOW LET $E = \underline{b}_1^{\perp}$ AND $\{\underline{e}_2, \dots, \underline{e}_n\}$ ANY ORTHONORMAL BASIS OF E (TO BE CHOSEN LATER).

LET $\mathcal{E} = \{ \underline{b}_1, \underline{e}_2, \dots, \underline{e}_n \}$, SO $\mathcal{E}$ IS AN ORTHONORMAL BASIS OF $V = \mathbb{C}^n$.

$$\Rightarrow [A]_{\mathcal{E}} = \left(\begin{array}{c|c} \lambda_1 & ? \\ \hline 0 & \\ \vdots & A_1 \\ 0 & \end{array} \right)$$

WHERE $A_1 \in M_{n-1, n-1}(\mathbb{C})$.

BY INDUCTION HYPOTHESIS, A_1 IS TRIANGULAR, SO

$[A]_{\mathcal{E}}$ IS TRIANGULAR AND $A = UTU^{-1}$

WITH $T = \left(\begin{array}{c|c} \lambda_1 & \\ \hline 0 & A_1 \end{array} \right)$ AND $U = (\underline{b}_1, \underline{e}_2, \dots, \underline{e}_n)$.

FINALLY, WE USE THIS DECOMPOSITION TO SHOW:

<u>THM.</u> (COMPLEX SPECTRAL THM.)

$$N \text{ NORMAL} \iff N = U D U^{-1}$$

ORTHONORMAL BASIS OF E-VECTORS OF N.

DIAGONAL MATRIX OF E-VALUES OF N (POSSIBLY COMPLEX)

Pf. "$\Rightarrow$" N HAS SCHUR DECOMPOSITION

$$N = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{array}\right)$$

IN SOME ORTHONORMAL BASIS $\hat{\mathcal{E}}$, WHERE N_1 IS AN UPPER TRIANGULAR MATRIX.

$$(N^*N)_{11} = \bar{a}_{11} a_{11} = |a_{11}|^2.$$

$$\shortparallel$$

$$(NN^*)_{11} = a_{11}\bar{a}_{11} + \cdots + a_{1n}\bar{a}_{1n} = |a_{11}|^2 + \cdots + |a_{1n}|^2$$

$$\Rightarrow \quad a_{12} = \cdots = a_{1n} = 0.$$

SO, $N = \left(\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & N_1 \end{array}\right)$ AND $N^*N = NN^*$

$$\Rightarrow \quad N_1^* N_1 = N_1 N_1^*.$$

REPEATING THE SAME STEPS SHOWS THAT $N = \left(\begin{array}{cc|c} a_{11} & & 0 \\ & a_{12} & \\ \hline & 0 & N_2 \end{array}\right)$,

SO THIS FINALLY GIVES THAT $N = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$

IN ORTHONORMAL BASIS $\mathcal{E}$.

"$\Leftarrow$" $N = UDU^{-1}$ WITH $U = (\underline{e}_1, \dots, \underline{e}_n)$

$\Rightarrow [N]_{\mathcal{E}} = D$ FOR $\mathcal{E} = \{\underline{e}_1, \dots, \underline{e}_n\}$.

$\Rightarrow [N^*]_{\mathcal{E}} = D^*$

THEN, $D^*D = DD^* \Rightarrow [N^*N]_{\mathcal{E}} = [NN^*]_{\mathcal{E}}$

$\Rightarrow N$ NORMAL.