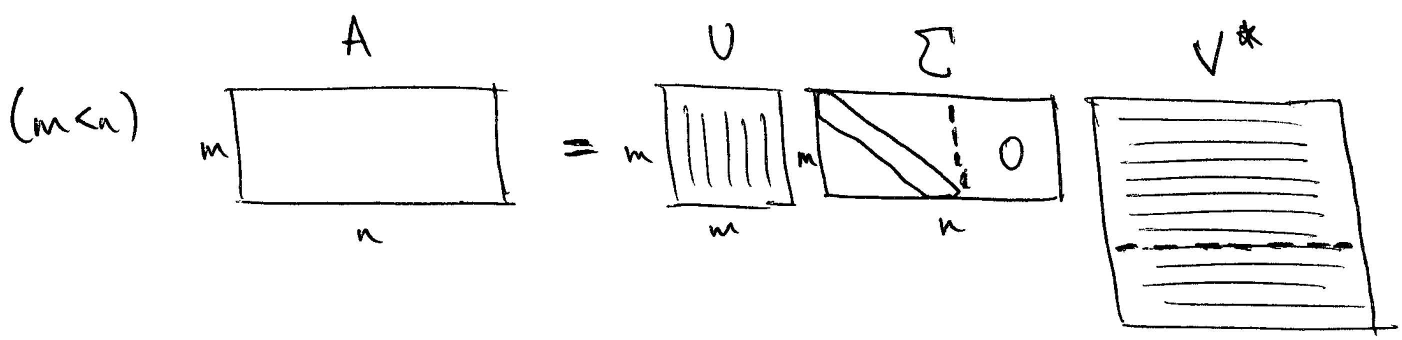
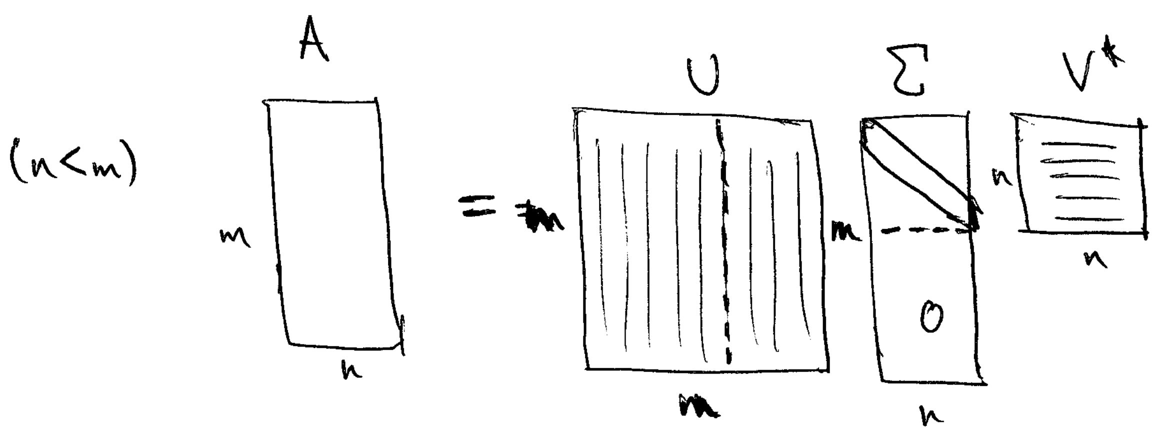
Lecture 34
04/16/12

Singular value decomposition (SVD) (cont'd):

Last time, we introduced SVD for any square matrices — in fact, can find SVD of <u>any</u> $A \in M_{m,n}(\mathbb{C})$

m,n may be different!





$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ (rectangular diagonal matrix w/ singular values on diagonal)

$V \in M_{n,n}(\mathbb{C})$ unitary (e-vectors of A^*A)

$U \in M_{m,m}(\mathbb{C})$ unitary (e-vectors of AA^*)

PROCEDURE TO FIND FIND SVD EXACTLY SAME AS BEFORE, EXCEPT

(4') IF NECESSARY: $\underline{v}_{r+1}, \dots, \underline{v}_n \in \mathrm{Ker}(A^*A)$

$\underline{u}_{r+1}, \dots, \underline{u}_m \in \mathrm{Ker}(AA^*)$

FIND USING GRAM-SCHMIDT.

<u>EX.</u> FIND SVD OF $A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}$

(1) $A^*A = \begin{pmatrix} 9 & 9 \\ 8 & 8 \end{pmatrix} \Rightarrow \lambda_1 = 17, \ \lambda_2 = 1$

W/ CORRESPONDING EVECTORS

$\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \underline{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

ORTHONORMAL.

(2) $\sigma_1 = \sqrt{\lambda_1} = \sqrt{17}, \ \sigma_2 = \sqrt{\lambda_2} = 1.$

(3) $\underline{u}_1 = \frac{1}{\sigma_1} A\underline{v}_1 = \frac{1}{\sqrt{17}\sqrt{2}}\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{34}}\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$

$\underline{u}_2 = \frac{1}{\sigma_2} A\underline{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

(4) STILL NEED $\underline{u}_3$. NOTE THAT

$AA^* = \begin{pmatrix} 5 & 6 & 4 \\ 6 & 8 & 6 \\ 4 & 6 & 5 \end{pmatrix} \xrightarrow{\text{ROW REDUCE}} \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \operatorname{Ker}(AA^*) = \text{SPAN}\left\{\frac{1}{\sqrt{17}}(2,-3,2)^T\right\}$$

$$\Rightarrow \underline{u}_3 = \frac{1}{\sqrt{17}}(2,-3,2)^T .$$

(EASY TO CHECK THAT $\underline{u}_3 \perp \underline{u}_1$ AND $\underline{u}_3 \perp \underline{u}_2$).

SO,

$$A = U \Sigma V^*$$

$$U = \begin{pmatrix} \frac{3}{\sqrt{34}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix}, \quad V^* = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

APPLICATIONS OF SVD:

IT IS EASY TO SEE THAT $A = U\Sigma V^*$

$$\Rightarrow A = \sum_{i=1}^{r} \sigma_i \underline{u}_i \underline{v}_i^* \qquad (r = \text{RANK}(A)).$$

NOTE THAT $\underline{u}_i \underline{v}_i^*$ IS A RANK 1 MATRIX (WHY?),

SO $A = \sum_{i=1}^{p} \sigma_i \underline{u}_i \underline{v}_i^*$ IS A RANK p MATRIX.

AS WE NOW SEE, THIS IS THE "BEST" p-RANK APPROXIMATION TO A IN SOME SENSE.

WE NEED A NOTION OF DISTANCE FOR MATRICES.

CONSIDER THE FROBENIUS INNER PRODUCT ON $V = M_{m,n}(\mathbb{C})$:

$$\langle A | B \rangle = \mathrm{Tr}(A^* B).$$

$$\Rightarrow \|A\|^2 = \langle A | A \rangle = \mathrm{Tr}(A^* A) = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2.$$

CAN CHECK THAT FROBENIUS NORM IS UNITARILY INVARIANT — I.E.,

$$W \text{ ISOMETRY} \Rightarrow \begin{array}{l} \|WA\| = \|A\| \\ \|AW\| = \|A\| \end{array} \text{ FOR ANY } A.$$

<u>THM.</u> IF $A = U \Sigma V^*$, THE CLOSEST (IN SQUARED FROBENIUS NORM!) p-RANK APPROXIMATION TO A IS $B = U \Sigma_p V^*$, WHERE $\Sigma_p = \mathrm{DIAG}(\sigma_1, \ldots, \sigma_p)$

<u>PF.</u> WE SEEK TO SOLVE

$$\begin{cases} \text{MINIMIZE } \|A - B\|^2 \text{ OVER ALL } B \text{ SUCH THAT} \\ \mathrm{RANK}(B) \leq p \leq r. \end{cases}$$

IF A HAS SVD $A = U \Sigma V^*$, DEFINE A NEW MATRIX $S = U^* B V$, SO THAT $B = U S V^*$.

THEN,

$$\begin{aligned} \|A - B\|^2 &= \|U \Sigma V^* - U S V^*\|^2 \\ &= \|U (\Sigma - S) V^*\|^2 \end{aligned}$$

$$= \|\Sigma - S\|^2 \quad (\text{SINCE } U, V^* \text{ ISOMETRIES})$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} |\Sigma_{ij} - S_{ij}|^2$$

$$= \sum_{i=1}^{r} |\sigma_i - S_{ii}|^2 + \sum_{\substack{i,j \text{ s.t.} \\ \Sigma_{ij}=0}} \sum |S_{ij}|^2 .$$

- TO MAKE THE 2ND TERM AS SMALL ~~AS~~ POSSIBLE, WE TAKE $S_{ij} = 0$ EVERYWHERE $\Sigma_{ij} = 0$. SO, AT THIS POINT WE KNOW THAT $S = \text{DIAG}(S_{11}, S_{22}, \dots, S_{rr})$.

- TO MAKE THE 1ST TERM AS SMALL AS POSSIBLE, WE TAKE $S_{ii} = \sigma_i$ FOR $i = 1, \dots, p$, AND ZERO OTHERWISE. THIS IS THE BEST S OF RANK $\leq p$ WHICH MINIMIZES $\|\Sigma - S\|^2$.

SO, $S = \text{DIAG}(\sigma_1, \dots, \sigma_p) \implies B = U\Sigma_p V^*$ IS THE BEST p-RANK APPROX!

THIS PROPERTY HAS WIDE APPLICABILITY TO DATA COMPRESSION, PRINCIPAL COMPONENT ANALYSIS, FACTOR ANALYSIS, PATTERN RECOGNITION, ...

EX. $$A = \begin{pmatrix} 1.01 & 1 & 1 \\ 1 & 1.01 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

RANK $(A) = 3$, BUT IT SHOULD BE VERY CLOSELY APPROX. BY A RANK 1 MATRIX SINCE $1.01 \approx 1$. WE FIND

$$\Sigma = \begin{pmatrix} \overset{\sigma_1}{\overset{\shortparallel}{3.01}} & & \\ & \overset{\sigma_2}{\overset{\shortparallel}{0.01}} & \\ & & \overset{\sigma_3}{\overset{\shortparallel}{0.01}} \end{pmatrix}$$ SO $\sigma_1 \gg \sigma_2, \sigma_3$.

$\rightsquigarrow$ RANK $(A) = 3$ BUT IS VERY "CLOSE" TO RANK 1 MATRIX

$$\sigma_1 \underline{u}_1 \underline{v}_1^* = \begin{pmatrix} 1.01 & 1.01 & 1.01 \\ 1.01 & 1.01 & 1.01 \\ 1.01 & 1.01 & 1.01 \end{pmatrix}$$

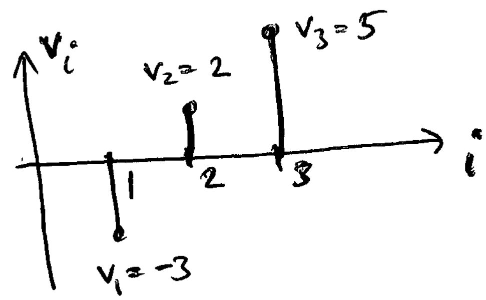
LECTURE 35
04/18/12

1

INFINITE-DIM VECTOR SPACES, INNER PROD. SPACES (6.8):

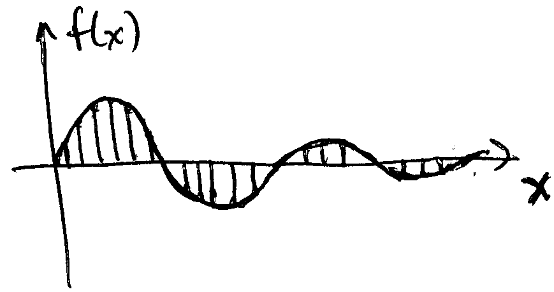
WE CAN THINK OF FUNCTIONS AS INF.-DIM. VECTORS, BY THE FOLLOWING ANALOGY:

$$\underline{v} = (v_1, v_2, v_3)^T = \{v_i\}_{i=1}^{3} \in \mathbb{R}^3, \quad \text{FOR EX.,} \quad \underline{v} = (-3, 2, 5)^T$$



ANALOGOUSLY,

$$f = \{f(x)\}_{x \in \mathbb{R}} \in V, \quad \text{FOR EX.,} \quad f(x) = e^{-x} \sin x$$



<u>MOTIVATION</u>: INFINITE-DIM. SPACES ARISE IN MANY SETTINGS, BUT ARE ESPECIALLY IMPORTANT WHEN CONSIDERING PARTIAL DIFFERENTIAL EQUATIONS (PDE).

RECALL THAT FOR LINEAR, CONST-COEFF., HOMOGENEOUS ODE WE CONSIDERED

$$\begin{cases} \dfrac{d\underline{v}}{dt} = A\underline{v}, \\ \underline{v}(0) \text{ GIVEN} \end{cases} \quad \text{WHERE } \underline{v} = \underline{v}(t) \in \mathbb{R}^n \text{ FOR ALL } t \geq 0. \quad \text{AND } A: \mathbb{R}^n \to \mathbb{R}^n \text{ LINEAR OPERATOR.}$$

SIMILARLY, LINEAR, CONST. COEFF., HOMOGENEOUS PDE TAKE THE FORM

$$\begin{cases} \dfrac{du}{dt} = Lu \\ u(t=0) \text{ GIVEN} \end{cases}, \quad \text{WHERE } u = u(t) \in V \text{ FOR ALL } t \geq 0 \text{ AND } L : V \to V \text{ LINEAR OPERATOR.}$$

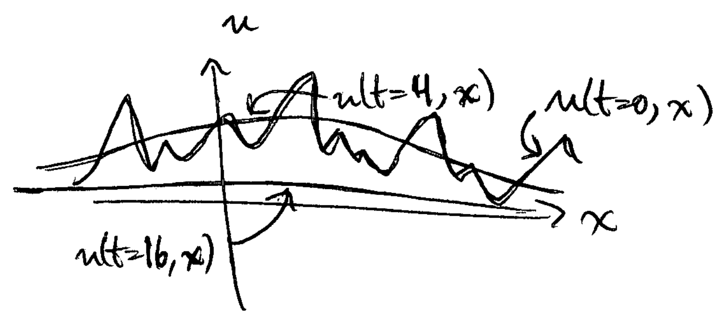
HERE, V IS A <u>FUNCTION SPACE</u> (I.E., A SPACE IN WHICH THE TYPICAL ELEMENT IS A FUNCTION $f(x)$, $x \in \mathbb{R}$).

THEREFORE, WE SHOULD THINK OF $u(t) \doteq u(t, x)$ FOR $x \in \mathbb{R}$ AND L AS AN OPERATOR ON FUNCTIONS OF x. THAT IS,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = L\, u(x,t) \\ u(0, x) \text{ GIVEN} \end{cases}.$$

FOR EX., IF $L = \dfrac{\partial^2}{\partial x^2}$ THIS IS A CLASSICAL PDE CALLED THE <u>HEAT EQN.</u>:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0, x) \text{ GIVEN} \end{cases} \longrightarrow$$



<u>WARNING:</u> WHILE MANY OF THE CONCEPTS WE HAVE LEARNED FOR FINITE-DIM. SPACES WILL CARRY OVER TO ∞-DIM. SPACES, MANY WILL NOT. ISSUES WILL TYPICALLY ARISE DUE TO A LACK OF CONVERGENCE OF SOME SUM.

FOR EX., CONSIDER ANY TWO FINITE-DIM $n \times n$ MATRICES A_n AND B_n.

THEN $\mathrm{Tr}(A_n B_n) = \mathrm{Tr}(B_n A_n) \Rightarrow \mathrm{Tr}(A_n B_n - B_n A_n) = 0$

FOR ANY $A_n, B_n \in M_{n,n}(\mathbb{C})$.

HOWEVER, IF

$$A_\infty = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \vdots \\ & & \ddots & \ddots \\ 0 & \cdots & & \end{pmatrix}, \quad B_\infty = A_\infty^T = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \vdots \\ & 1 & \ddots & \\ 0 & \cdots & \ddots & \end{pmatrix}$$

INFINITE-DIM MATRICES.

THEN $A_\infty B_\infty = \begin{pmatrix} 1 & & 0 \\ & 1 & \vdots \\ 0 & \cdots & \ddots \end{pmatrix}$, $B_\infty A_\infty = \begin{pmatrix} 0 & & 0 \\ & 1 & \vdots \\ & & 1 \\ 0 & \cdots & \ddots \end{pmatrix}$

$$\Rightarrow A_\infty B_\infty - B_\infty A_\infty = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ & & 0 \\ 0 & & \ddots \end{pmatrix} \Rightarrow \mathrm{Tr}(A_\infty B_\infty - B_\infty A_\infty) = 1 \neq 0!$$

WE SEE THAT $\lim_{n\to\infty} \mathrm{Tr}(A_n B_n - B_n A_n) \neq \mathrm{Tr}(\lim_{n\to\infty}(A_n B_n - B_n A_n))$

WHERE $A_n = \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}}_{n} \Big\} n$, $B_n = A_n^T = \underbrace{\begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}}_{n} \Big\} n$.

THE REASON FOR THIS FAILURE IS THAT $\mathrm{Tr}(A_\infty B_\infty)$ AND $\mathrm{Tr}(B_\infty A_\infty)$ ARE BOTH ∞! SO WE SHOULD EXPECT SOME TROUBLE BECAUSE OF A LACK OF CONVERGENCE.

EXAMPLES OF ∞-DIM VECTOR SPACES:

(1) $V = \ell_2(\mathbb{R})$ (SPACE OF SQUARE-SUMMABLE SEQUENCES IN $\mathbb{R}$)

DEF. $\underline{v} = (v_1, v_2, v_3, \dots) \in \ell_2(\mathbb{R})$ IF $\sum_{i=1}^{\infty} |v_i|^2 < \infty$, $v_i \in \mathbb{R}$.

- $\ell_2(\mathbb{R})$ IS A VECTOR SPACE SINCE

(i) $\underline{v}, \underline{w} \in \ell_2(\mathbb{R}) \Rightarrow \underset{\underset{(v_1+w_1,\, v_2+w_2,\, \dots)}{\shortparallel}}{\underline{v}+\underline{w}} \in \ell_2(\mathbb{R})$

PF.

$$\sum_{i=1}^{\infty} |v_i+w_i|^2 = \sum_{i=1}^{\infty} |v_i|^2 + |w_i|^2 + \underbrace{2|v_i||w_i|}_{\leq |v_i|^2+|w_i|^2} < \infty .$$

(SINCE FOR ANY $a, b \in \mathbb{R}$, $(|a|-|b|)^2 \geq 0$)

(ii) $\underline{v} \in \ell_2(\mathbb{R})$, c SCALAR $\Rightarrow c\underline{v} \in \ell_2(\mathbb{R})$.

PF. EASY.

- $\mathcal{E} = \{\underline{e}_i\}_{i=1}^{\infty}$, $\underline{e}_i = (0, \dots, 0, 1, 0, \dots)$ IS THE STANDARD BASIS OF $\ell_2(\mathbb{R})$. NOTE THAT ANY $\underline{v} \in \ell_2(\mathbb{R})$ CAN BE WRITTEN AS AN INFINITE LINEAR COMBINATION OF THE $\{\underline{e}_i\}$.

  ith COORDINATE

- $\ell_2(\mathbb{R})$ IS AN INNER PRODUCT SPACE WITH $\langle \underline{v} | \underline{w} \rangle = \sum_{i=1}^{\infty} v_i w_i$. NOTE THAT THIS IS A GENERALIZATION OF THE INNER PRODUCT OF $\mathbb{R}^n$ AS $n \to \infty$.

- SIMILARLY, ONE CAN DEFINE $\ell_2(\mathbb{C})$ AS ALL $\underline{v} = (v_1, v_2, v_3, \ldots)$ S.T. $v_i \in \mathbb{C}$ AND $\sum_{i=1}^{\infty} |v_i|^2 < \infty$.

  THE INNER PRODUCT IS THEN $\langle \underline{v} | \underline{w} \rangle = \sum_{i=1}^{\infty} \overline{v_i} w_i$.

(2) $V = L_2(U)$ (SPACE OF SQUARE-INTEGRABLE FUNCTIONS $f : U \to \mathbb{R}$).

__DEF.__ $f \in L_2(U)$ IF $f : U \to \mathbb{R}$, AND $\int_U |f(x)|^2 \, dx < \infty$.

TYPICALLY, WE TAKE $U = [0, a]$ FOR SOME $a > 0$ OR $U = \mathbb{R}$.

- $L_2(U)$ IS A VECTOR SPACE, SINCE

  (i) $f, g \in L_2(U) \Rightarrow f+g \in L_2(U)$

  __PF.__
$$\int_U |f(x)+g(x)|^2 \, dx \le \int_U \Big(|f(x)|^2 + |g(x)|^2 + \underbrace{2|f(x)||g(x)|}_{\le |f(x)|^2 + |g(x)|^2}\Big) dx$$
$$< \infty .$$

  (ii) $cf \in L_2(U)$ FOR ANY SCALAR c.

  __PF.__ EASY.

- WHAT IS A BASIS FOR $L_2(U)$? WE WILL SEE LATER WHEN DISCUSSING FOURIER SERIES.

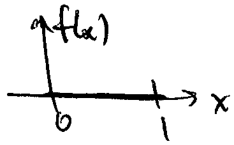
- $L_2(U)$ IS AN INNER PRODUCT SPACE WITH

$$\langle f | g \rangle = \int_U f(x) g(x)\, dx.$$

- SIMILARLY, ONE CAN DEFINE $L_2(U; \mathbb{C})$ AS ALL $f : U \to \mathbb{C}$ SUCH THAT $\int_U |f(x)|^2\, dx < \infty$.

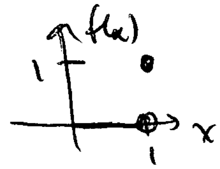
<u>WARNING:</u> THERE ARE MANY SUBTLETIES WHEN WORKING WITH $L_2(U)$ WHICH WE WILL TRY TO AVOID. FOR EX., NOTE THAT FOR $U = [0,1]$,

$$f = 0 \text{ FOR ALL } x \in [0,1] \Rightarrow \|f\|^2 = \langle f | f \rangle = \int_0^1 |f(x)|^2\, dx = 0,$$



AS EXPECTED. HOWEVER, WE ALSO SEE THAT

$$f(x) = \begin{cases} 0 & \text{FOR ALL } x \in [0,1) \\ 1 & \text{FOR } x = 1 \end{cases} \Rightarrow \|f\|^2 = \langle f | f \rangle = \int_0^1 |f(x)|^2\, dx = 0 \;!$$



THEREFORE, POSITIVITY OF THE INNER PRODUCT DOES NOT HOLD AS WE HAVE DEFINED IT SINCE THE INTEGRAL DOES NOT "SEE" INDIVIDUAL POINTS ON THE LINE. TO REMEDY THIS, WE WILL THINK OF THE ZERO ELEMENT AS <u>ANY</u> f S.T. $\int_U |f|\, dx = 0$, AND $f = g$ IN $L_2(U)$ IF $\int_U |f - g|\, dx = 0$.

LECTURE 36
04/20/12

Q: WHAT IS A BASIS OF $L^2(U)$ WHEN

(a) $U = [0, a]$, $a > 0$?

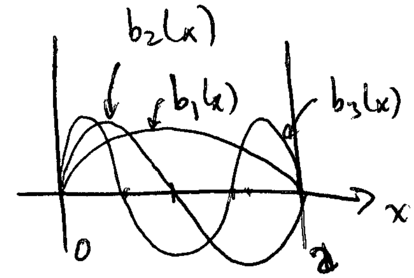
(b) $U = \mathbb{R}$?

AS WE WILL SEE, THE ANSWER IN CASE (a) IS SIGNIFICANTLY SIMPLER THAN IN CASE (b).

FOURIER SERIES ON AN INTERVAL (6.9):

SUPPOSE $U = [0, a]$, $a > 0$.

LET $b_n(x) = \sin\left(\frac{n\pi x}{a}\right)$, $n = 1, 2, 3, \ldots$



- EASY TO CHECK THAT $b_n \in L_2([0,a])$ FOR EACH n.
- FURTHERMORE, $\{b_n\}_{n=1}^{\infty}$ ARE ORTHOGONAL!

PF. $$\langle b_m | b_n \rangle = \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx$$

$\left(\text{LETTING } y = \frac{x}{a}\right)$ $$= a \int_0^1 \sin(m\pi y) \sin(n\pi y)\, dy$$

$$= \frac{a}{2} \int_0^1 \left[\cos((n-m)\pi y) - \cos((n+m)\pi y)\right] dy$$

(USING THE IDENTITY $2\sin(u)\sin(v) = \cos(u-v) - \cos(u+v)$)

$$= \begin{cases} \frac{a}{2}\left[\frac{\sin((n-m)\pi y)}{(n-m)\pi} - \frac{\sin((n+m)\pi y)}{(n+m)\pi}\right]_{y=0}^{y=1}, & m \neq n \\ \frac{a}{2}\left[1 - \frac{\sin(2n\pi y)}{2n\pi}\right]_0^1, & m = n \end{cases}$$

$$= \begin{cases} 0 & \text{IF } m \neq n \\ \frac{a}{2} & \text{IF } m = n . \end{cases}$$

$\Rightarrow \{b_n\}_{n=1}^{\infty}$ ORTHOGONAL (<u>NOT</u> ORTHO<u>NORMAL</u>!). IN $L_2([0,a])$

- IN FACT, $\{b_n\}_{n=1}^{\infty}$ IS A BASIS OF $L_2([0,a])$.

"<u>PF.</u>" LET $L = \frac{d^2}{dx^2}$. THEN AS WE HAVE SHOWN PREVIOUSLY, L IS SELF-ADJOINT ON $L_2([0,a])$.

NOTE THAT $\{b_n\}_{n=1}^{\infty}$ ARE "EIGENVECTORS" (WE WILL CALL THEM <u>EIGENFUNCTIONS</u>) OF L W/ CORRESPONDING EIGENVALUES $\lambda_n = -\frac{n^2\pi^2}{a^2}$ SINCE $L\, b_n(x) = \lambda_n\, b_n(x)$.

ASSUMING THE SPECTRAL THM. HOLDS FOR SELF-ADJOINT OPERATORS ON ∞-DIM. SPACES, THIS IMPLIES THAT $\{b_n\}_{n=1}^{\infty}$ IS AN ORTHOGONAL <u>BASIS</u> OF $L_2([0,a])$.

(TECHNICALLY, L IS SELF-ADJOINT ON THE SPACE

$$C_0^\infty([0,a]) = \{ f : [0,a] \to \mathbb{R} : f(0) = f(a) = 0 \text{ AND } f \text{ INFINITELY DIFFERENTIABLE} \}$$

THIS, WITH THE FACT THAT FOR ANY $f \in L_2([0,a])$ THERE IS A SEQUENCE $\{g_m\}_{m=1}^\infty \subset C_0^\infty([0,a])$ THAT APPROXIMATES f (I.E., $\|f - g_m\|_{L_2([0,a])} = \int_0^a |f(x) - g_m(x)|^2 dx \xrightarrow[m\to\infty]{} 0.$)

IMPLIES THAT $\{b_n\}_{n=1}^\infty$ IS A BASIS OF $L_2([0,a])$.)

- NOTE THAT WE CAN NOW WRITE

$$f(x) = \sum_{n=1}^\infty c_n \, b_n(x) = \sum_{n=1}^\infty c_n \sin\left(\frac{n\pi x}{a}\right)$$

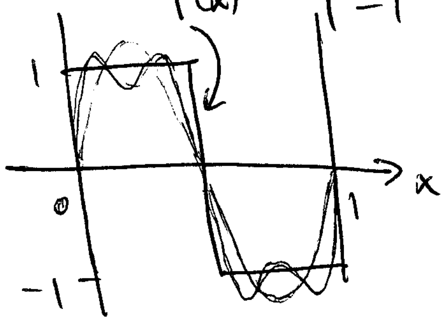
FOR ANY $f \in L_2([0,a])$. THE c_n's ARE CALLED <u>FOURIER COEFFICIENTS</u> OF f AND THE ABOVE REPRESENTATION IS CALLED <u>FOURIER SERIES</u>.

- TO FIND c_n, WE NOTE THAT

$$c_n = \frac{\langle b_n | f \rangle}{\langle b_n | b_n \rangle} = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

EX.

$$f(x) = \begin{cases} 1 & \text{IF } x \in [0, \tfrac{1}{2}) \\ -1 & \text{IF } x \in [\tfrac{1}{2}, 1] \end{cases}.$$



$$\Rightarrow \quad c_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= 2 \left(\int_0^{\frac{1}{2}} \sin(n\pi x)\, dx - \int_{\frac{1}{2}}^{1} \sin(n\pi x)\, dx \right)$$

$$= \frac{2}{n\pi} \left(\Big[-\cos(n\pi x) \Big]_0^{\frac{1}{2}} + \Big[\cos(n\pi x) \Big]_{\frac{1}{2}}^{1} \right)$$

$$= \frac{2}{n\pi} \left(1 - 2\cos\left(\frac{n\pi}{2}\right) + \cos(n\pi) \right),$$

$$n = 1, 2, 3, \ldots$$

NOTE THAT $|c_n| \sim \frac{1}{n}$ AS $n \to \infty$.

WITH ONLY A FEW TERMS OF THE SERIES WE CAN GET A FAIRLY GOOD APPROXIMATION TO OUR ORIGINAL FUNCTION — I.E.

$$f(x) \approx \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{a}\right).$$