LECTURE 37
04/23/12

RECALL THE FOURIER BASIS $\{b_n\}_{n=1}^{\infty}$ OF $L_2([0,a])$ GIVEN BY $b_n(x) = \sin\left(\frac{n\pi x}{a}\right)$, $n = 1, 2, 3, \ldots$. ANY FUNCTION $f \in L_2([0,a])$ CAN BE WRITTEN AS $f(x) = \sum_{n=1}^{\infty} c_n b_n(x)$, WHERE

$$c_n = \frac{\langle b_n | f \rangle}{\langle b_n | b_n \rangle} = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx .$$

NOTE: THE FOURIER COEFFICIENTS c_n TELL US "HOW MUCH" OF THE TERM $\sin\left(\frac{n\pi x}{a}\right)$ IS PRESENT IN $f(x)$ — IN PARTICULAR, NOTE THAT $c_n \to 0$ AS $n \to \infty$ (OTHERWISE THE SERIES COULD NOT CONVERGE!) IN ADDITION, THE RATE OF DECAY OF THE c_n'S AS $n \to \infty$ DEPENDS ON HOW SMOOTH f IS SINCE LARGE DERIVATIVES CAN ONLY BE CAPTURED BY HIGH FREQUENCY (I.E, LARGE n) TERMS.

$|f'|$ LARGE $\Rightarrow$ HIGH FREQUENCIES IMPORTANT

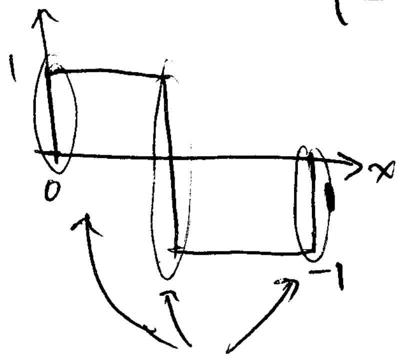
$\Rightarrow$ $c_n \to 0$ (AS $n \to \infty$) <u>SLOWLY</u>

$|f'|$ SMALL $\Rightarrow$ HIGH FREQUENCIES UNIMPORTANT

$\Rightarrow$ $c_n \to 0$ (AS $n \to \infty$) <u>QUICKLY</u>.

Ex.

$$f(x) = \begin{cases} 1, & x \in [0, \tfrac{1}{2}) \\ -1, & x \in [\tfrac{1}{2}, 1] \end{cases}$$



$|f'| \gg 1.$

WE SAW LAST TIME THAT

$$c_n = \frac{2}{n\pi}\left(1 - 2\cos\left(\frac{n\pi}{2}\right) + \cos(n\pi)\right)$$

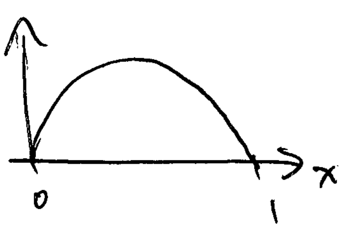
$$\sim \frac{1}{n} \quad \text{AS} \quad n \to \infty.$$

" PROPORTIONAL TO "

$\Rightarrow$ SLOW DECAY DUE TO LARGE DERIVATIVES $|f'|$ AT $x = 0, \frac{1}{2}, 1$.

Ex.

$$f(x) = x - x^2, \quad x \in [0, 1].$$



CALCULATING THE COEFFICIENTS WE GET

$$c_n = 2\int_0^1 (x - x^2)\sin(n\pi x)\,dx$$

$$= \frac{4(1 - \cos(n\pi))}{n^3\pi^3}$$

$$\sim \frac{1}{n^3} \quad \text{AS} \quad n \to \infty$$

$\Rightarrow$ FAST DECAY SINCE $|f'|$ IS EVERYWHERE BOUNDED BY 2.

PROPERTIES :

(i) BASIS $\mathcal{B} = \{b_n\}_{n=1}^{\infty}$ MAKES $L_2([0,a])$ LOOK LIKE ℓ_2 !

IF $f, g \in L_2([0,a])$, $f(x) = \sum_{n=1}^{\infty} c_n b_n(x)$, $g(x) = \sum_{n=1}^{\infty} d_n b_n(x)$,

$$\langle f | g \rangle_{L_2} = \left\langle \sum_{n=1}^{\infty} c_n b_n \,\middle|\, \sum_{n=1}^{\infty} d_m b_m \right\rangle_{L_2}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n d_m \langle b_n | b_m \rangle_{L_2}$$

$$= \frac{a}{2} \sum_{n=1}^{\infty} c_n d_n = \frac{a}{2} \langle [f]_{\mathcal{B}} | [g]_{\mathcal{B}} \rangle_{\ell_2}$$

SINCE $[f]_{\mathcal{B}} = (c_1, c_2, \dots) \in \ell_2$.

$[g]_{\mathcal{B}} = (d_1, d_2, \dots) \in \ell_2$.

$$\Rightarrow \boxed{\|f\|^2_{L_2([0,a])} = \frac{a}{2} \| [f]_{\mathcal{B}} \|^2_{\ell_2}}$$

(PARSEVAL'S THM.).

IN PARTICULAR, THIS QUANTIFIES THE ERROR BETWEEN A FUNCTION AND ITS FOURIER SERIES (TRUNCATED AT N^{th} TERM) AS

$$\text{ERROR}^2 = \|f - f_N\|^2_{L_2} = \left\| \sum_{n=N+1}^{\infty} c_n b_n \right\|^2_{L_2} = \frac{a}{2} \sum_{n=N+1}^{\infty} |c_n|^2.$$

(ii) FOURIER SERIES $f = \sum c_n b_n$ IS IDEAL WHEN WORKING WITH EVEN DERIVATIVES!

INDEED, RECALL THAT b_n IS THE nTH EIGENFUNCTION OF $L = \frac{d^2}{dx^2}$ w/ CORRESPONDING EIGENVALUE $\lambda_n = \frac{-n^2\pi^2}{a^2}$.

THEN,

$$
\begin{aligned}
f''(x) = Lf(x) &= L\left(\sum_{n=1}^{\infty} c_n b_n(x)\right) \\
&= \sum_{n=1}^{\infty} c_n L b_n(x) \\
&= \sum_{n=1}^{\infty} c_n \lambda_n b_n(x).
\end{aligned}
$$

SO, THE nTH FOURIER COEFFICIENT OF f'' IS SIMPLY $\lambda_n c_n = -\frac{n^2\pi^2}{a^2} c_n$ (I.E., DERIVATIVES LEAD TO A SIMPLE MULTIPLICATION IN TERMS OF FOURIER COEFFICIENTS!

<u>EX.</u> CONSIDER THE PARTIAL DIFFERENTIAL EQUATION (PDE)

$$
\begin{cases}
\partial_t u = \partial_{xx} u \rightsquigarrow \text{“HEAT EQN.”} \\
u(0,t) = u(a,t) = 0 \quad \text{(BOUNDARY CONDITIONS)} \\
u(x,0) \text{ GIVEN.} \quad \text{(INITIAL CONDITIONS)}
\end{cases}
$$

HERE, $u(x,t)$ IS TO BE THOUGHT OF AS THE TEMPERATURE OF A THIN WIRE AT LOCATION $x \in [0,a]$ AT TIME $t \geq 0$.

THE BOUNDARY CONDITIONS $u(0,t) = u(a,t) = 0$ IMPLY THAT AT ALL TIMES THERE IS NO HEAT AT THE BOUNDARIES OF THE WIRE. SINCE HEAT SPREADS, WE SHOULD EXPECT THE SOLUTION $u(x,t) \to 0$ FOR ALL $x \in [0,a]$ AS $t \to \infty$ SINCE HEAT IS SUCKED OUT OF THE WIRE AT THE BOUNDARIES.

WE FIND THE SOLUTION BY SOLVING FOR THE EVOLUTION OF FOURIER COEFF.:

ASSUME $$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\, b_n(x)$$ (SEPARATION OF VARIABLES)

TIME-DEPENDENT FOURIER COEFF. ($c_n(t)$) — FOURIER BASIS ($b_n(x)$)

$$\Rightarrow \quad \partial_t u(x,t) = \sum_{n=1}^{\infty} \left(\frac{d}{dt} c_n(t)\right) b_n(x)$$

$$\partial_{xx} u(x,t) = \sum_{n=1}^{\infty} c_n(t)\, L\, b_n(x) \qquad \left(L = \frac{d^2}{dx^2}\right)$$

$$= \sum_{n=1}^{\infty} \left(\lambda_n c_n(t)\right) b_n(x)$$

$$\Rightarrow \quad \frac{d}{dt} c_n(t) = \lambda_n c_n(t) \quad \text{FOR ALL } n = 1,2,3,\dots$$

$$\Rightarrow \quad c_n(t) = c_n(0)\, e^{\lambda_n t} = c_n(0)\, e^{-\left(\frac{n^2\pi^2}{a^2}\right)t}, \quad n = 1,2,3,\dots$$

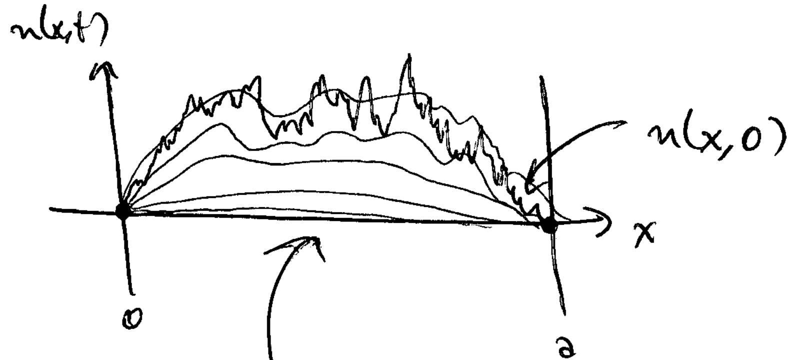
WHERE $u(x,0) = \sum_{n=1}^{\infty} c_n(0)\, b_n(x)$ __GIVEN__.

$$c_n(0) = \frac{2}{a}\int_0^a u(x,0)\, b_n(x)\, dx.$$

NOTE THAT $c_n(t) \to 0$ AS $t \to \infty$ FOR EVERY n,

BUT THAT THE DECAY IS <u>SIGNIFICANTLY</u> FASTER FOR LARGER FREQUENCIES n!

$\Rightarrow$ SOLUTION GETS <u>SMOOTHER</u> AS $t \to \infty$ AND GOES TO 0 EVERYWHERE.



SOLUTION GETS SMOOTHER IN x AS TIME PROGRESSES, GOES TO 0 AS $t \to \infty$.

<u>NOTE:</u> THIS IS THE ∞-DIM ANALOGUE OF CONTINUOUS-TIME EVOLUTION IN $\mathbb{R}^n$!

$$\begin{cases} \frac{d\underline{u}}{dt} = A\underline{u} \\ u(0) \text{ given} \end{cases}, \quad A : \mathbb{R}^n \to \mathbb{R}^n .$$

WRITING $\underline{u}(t) = \sum_{i=1}^{n} c_i(t)\, \underline{b}_i$ i^{th} E-VECTOR OF A

WE FOUND $c_i(t) = c_i(0)\, e^{\lambda_i t}$. i^{th} E-VALUE OF A

- <u>STABILITY</u> OF i^{th} MODE $\underline{b}_i$ WAS DETERMINED BY $Re(\lambda_i)$.

LECTURE 38
04/27/12

Q: WE HAVE FOUND ONE BASIS OF $L_2([0,a])$.
ARE THERE OTHERS, AND HOW TO FIND THEM?

FOURIER SERIES (CONT'D) (8.5, 8.7)

LET US CAREFULLY REDERIVE THE BASIS WE HAVE WORKED WITH SO FAR:

① FOURIER SINE BASIS: DEFINE THE SPACE

$$L_2^{DIR}([0,a]) = \{ f \in L_2([0,a]) : \overbrace{f(0)=0,\ f(a)=0}^{\text{"DIRICHLET BOUNDARY CONDITIONS"}},\ f \text{ INFINITELY DIFFERENTIABLE.} \}$$

NOTE THAT $L_2^{DIR}([0,a]) \subseteq L_2([0,a])$.

- IN FACT, THIS SUBSPACE "WELL-APPROXIMATES" $L_2([0,a])$ IN THAT FOR ANY $f \in L_2([0,a])$, THERE IS A SEQUENCE $\{f_n\}_{n=1}^{\infty} \subseteq L_2^{DIR}([0,a])$ SUCH THAT $\|f-f_n\|_{L_2}^2 \xrightarrow{n\to\infty} 0$ (I.E, WE CAN APPROX. f WITHIN AN ARBITRARILY SMALL ERROR BY AN ELEMENT OF $L_2^{DIR}([0,a])$.

- $L = \frac{d^2}{dx^2}$ IS SELF-ADJOINT ON $L_2^{DIR}([0,a])$.

PF. $\langle f | L g \rangle = \int_0^a f(x)\, g''(x)\, dx$

$f, g \in L_2^{DIR}([0,a])$

$$= \underbrace{\cancel{\left[f(x) g'(x) \right]_0^a}}_{\substack{=0 \text{ SINCE} \\ f(0)=0,\ f(a)=0}} - \int_0^a f'(x)\, g'(x)\, dx$$

$$= \underbrace{\cancel{\left[-f'(x) g(x) \right]_0^a}}_{\substack{=0 \text{ SINCE} \\ g(0)=0,\ g(a)=0}} + \int_0^a f''(x)\, g(x)\, dx$$

$$= \langle L f | g \rangle . \quad \checkmark$$

- WHAT ARE EIGENFUNCTIONS OF L ON $L_2^{DIR}([0,a])$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{DIR}([0,a]) \end{cases} \Longrightarrow \begin{cases} f''(x) = \lambda f(x) \\ f(0) = 0, \ f(a) = 0 \end{cases}$$

$$\Longrightarrow \quad f(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, 3, \ldots$$

$$\text{w/ } \lambda = -\frac{n^2\pi^2}{a^2} .$$

THIS GIVES THE ~~ORTHOGONAL~~ ORTHOGONAL BASIS (BY SELF-ADJOINTNESS OF L) $\{b_n\}_{n=1}^{\infty}$ ON $L_2^{DIR}([0,a])$ (AND THUS ON $L_2([0,a])$)

$$\boxed{\begin{array}{l} b_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, 3, \ldots \\ \text{w/ EIGENVALUES} \quad \lambda_n = \frac{-n^2\pi^2}{a^2}. \\ \boxed{\left(\text{FOURIER } \underline{\text{SINE}} \text{ BASIS}\right)} \end{array}}$$

$$\rightsquigarrow f(x) = \sum_{n=1}^{\infty} c_n b_n(x), \quad c_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx.$$

(2) STANDARD FOURIER BASIS (REAL): DEFINE THE SPACE

$$L_2^{PER}([0,a]) = \{ f \in L_2([0,a]) : \underbrace{f(0) = f(a),\ f'(0) = f'(a)}_{\text{"PERIODIC BOUNDARY CONDITIONS"}},\ f \text{ INFINITELY DIFFERENTIABLE} \}$$

NOTE THAT $L_2^{PER}([0,a]) \subseteq L_2([0,a])$.

- IN FACT, AS WITH $L_2^{DIR}([0,a])$, WE HAVE THAT $L_2^{PER}([0,a])$ WELL-APPROXIMATES $L_2([0,a])$.
- $L = \frac{d^2}{dx^2}$ IS SELF-ADJOINT ON $L_2^{PER}([0,a])$.

PF. $\langle f | L g \rangle = \int_0^a f(x)\, g''(x)\, dx$

$f, g \in L_2^{PER}([0,a])$.

$$= \underbrace{\cancel{\left[f(x)\, g'(x)\right]_0^a}}_{\substack{=0 \text{ since } f(0)=f(a), \\ g'(0)=g'(a)\,!}} - \int_0^a f'(x)\, g'(x)\, dx$$

$$= \underbrace{\cancel{\left[-f(x)\, g'(x)\right]_0^a}}_{\substack{=0 \text{ since } f'(0)=f'(a) \\ g(0)=g(a)\,!}} + \int_0^a f''(x)\, g(x)\, dx$$

$$= \langle Lf \mid g \rangle .$$

✓

- WHAT ARE EIGENFUNCTIONS OF L ON $L_2^{PER}([0,a])$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{PER}([0,a]) \end{cases} \Rightarrow \begin{cases} f''(x) = \lambda f(x) \\ f(0) = f(a),\ f''(0) = f(a). \end{cases}$$

$$\Rightarrow \quad f(x) = 1 \quad \text{w/ } \lambda = 0 \quad \underline{\text{or}}$$

$$f(x) = \sin\left(\frac{2n\pi x}{a}\right) \quad \text{w/ } \lambda = -\frac{4n^2\pi^2}{a^2} \quad \underline{\text{or}}$$

$$f(x) = \cos\left(\frac{2n\pi x}{a}\right) \quad \text{w/ } \lambda = -\frac{4n^2\pi^2}{a^2}.$$

THIS GIVES THE ORTHONORMAL BASIS FOR $L_2^{PER}([0,a])$ (AND THUS FOR $L_2([0,a])$)

$$\left\{ 1, \left\{ \sin\left(\frac{2n\pi x}{a}\right)\right\}_{n=1}^{\infty}, \left\{ \cos\left(\frac{2n\pi x}{a}\right)\right\}_{n=1}^{\infty} \right\}$$

W/ E-VALUES

$$\lambda_0 = 0, \quad \lambda_n = \frac{-4n^2\pi^2}{a^2}\ ! \quad n = 1, 2, 3, \ldots$$

(STANDARD FOURIER BASIS)

THAT IS, EIGENVALUE λ_n HAS GEOMETRIC MULTIPLICITY 2 (FOR $n = 1, 2, 3, \ldots$), WITH CORRESPONDING EIGENSPACE

$$E_{\lambda_n} = \text{SPAN}\left\{ \sin\left(\frac{2n\pi x}{a}\right), \cos\left(\frac{2n\pi x}{a}\right)\right\}.$$

$$\rightsquigarrow f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}\left[\alpha_n \cos\left(\frac{2n\pi x}{a}\right) + \beta_n \sin\left(\frac{2n\pi x}{a}\right)\right],$$

"ALPHA" $\curvearrowright$ $\alpha_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{2n\pi x}{a}\right)dx,$

"BETA" $\curvearrowright$ $\beta_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{2n\pi x}{a}\right)dx.$

③ STANDARD FOURIER BASIS (COMPLEX):

CONSIDER THE SPACE OF COMPLEX - VALUED SQUARE - INTEGRABLE FUNCTIONS $L_2([0,a];\mathbb{C})$ W/ INNER PRODUCT

$$\langle f | g \rangle = \int_0^a \overline{f(x)}\, g(x)\, dx.$$

DEFINE THE SPACE

$$L_2^{PER}([0,a];\mathbb{C}) = \{ f \in L_2([0,a];\mathbb{C}) : f(0) = f(a),\ f \text{ INFINITELY DIFF.} \}$$

AS BEFORE, $L_2^{PER}([0,a];\mathbb{C}) \subseteq L_2([0,a];\mathbb{C})$ WELL-APPROXIMATES. THEN, WE FIND THAT:

- $L = -i \frac{d}{dx}$ IS SELF-ADJOINT ON $L_2^{PER}([0,a];\mathbb{C})$.

PF.

$$\langle f | L g \rangle = -i \int_0^a \overline{f(x)}\, g'(x)\, dx$$

$f,g \in L_2^{PER}([0,a];\mathbb{C})$

$$= \underbrace{\left[-i\, \overline{f(x)}\, g(x) \right]_0^a}_{=0 \text{ SINCE } \overline{f(0)} = \overline{f(a)},\ g(0) = g(a)} + i \int_0^a \overline{f'(x)}\, g(x)\, dx$$

$$= \int_0^a \overline{(-i f'(x))}\, g(x)\, dx$$

$$= \langle L f | g \rangle .$$

✓

- WHAT ARE THE EIGENFUNCTIONS OF L ON $L_2^{PER}([0,a];\mathbb{C})$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{PER}([0,a];\mathbb{C}) \end{cases} \Rightarrow \begin{cases} -i f'(x) = \lambda f(x) \\ f(0) = f(a) \end{cases}$$

$$\Rightarrow \quad f(x) = \exp\left(\frac{2\pi i n x}{a}\right), \quad n \in \{\ldots, -1, 0, 1, \ldots\} = \mathbb{Z}.$$

$$\text{w/ } \lambda_n = \frac{2\pi n}{a}.$$

THIS GIVES THE ORTHOGONAL BASIS FOR $L_2^{\text{PER}}([0,a]; \mathbb{C})$ (AND THUS FOR $L_2([0,a]; \mathbb{C})$)

$$\boxed{\left\{ \exp\left(\frac{2\pi i n x}{a}\right) \right\}_{n=-\infty}^{n=\infty} \quad \text{w/ E-VALUES } \lambda_n = \frac{2\pi n}{a}}$$

(STD. FOURIER BASIS (COMPLEX))

$$\rightsquigarrow \quad f(x) = \sum_{n=-\infty}^{\infty} \gamma_n \exp\left(\frac{2\pi i n x}{a}\right),$$

"GAMMA" $\longrightarrow$ $$\gamma_n = \frac{1}{a} \int_0^a f(x) \exp\left(\frac{-2\pi i n x}{a}\right) dx.$$

<u>NOTE:</u> THE STD. FOURIER BASIS IN THE REAL CASE IS A SPECIAL CASE OF WHAT WE NOW HAVE.

USING THAT

$$\exp\left(\frac{2\pi i n x}{a}\right) = \cos\left(\frac{2n\pi x}{a}\right) + i \sin\left(\frac{2n\pi x}{a}\right),$$

WE HAVE THAT FOR $n = 0, 1, 2, \ldots$

$$\alpha_n = \gamma_n + \gamma_{-n}, \quad \beta_n = +i(\gamma_n - \gamma_{-n}).$$