Lecture 39
04/30/12

We have been considering $L_2(U)$ with

(i) $U = [0,a]$

(ii) $U = \mathbb{R}$.

(i) led to Fourier series. What about (ii)?

Q: How do we find a basis of $L_2(U)$?

Let's try to extend what we did to find Fourier series:

First, is $L = -i\frac{d}{dx}$ self-adjoint on $L_2(\mathbb{R};\mathbb{C})$? Yes.

PF: Let $f, g \in L_2(\mathbb{R};\mathbb{C})$. Then,

$$\langle f | Lg \rangle = \int_{-\infty}^{\infty} \overline{f(x)}\,(-ig'(x))\,dx$$

$$= \underbrace{\cancel{\left[-i\overline{f(x)}\,g(x)\right]_{x=-\infty}^{x=+\infty}}}_{\substack{=0 \text{ since } f,g\in L_2(\mathbb{R};\mathbb{C}) \\ \Rightarrow f(x),\,g(x)\to 0 \text{ as } |x|\to\infty.}} + \underbrace{\int_{-\infty}^{\infty} \overline{(-if'(x))}\,g(x)\,dx}_{=\langle Lf | g\rangle}.$$

✓

- Does L have eigenfunctions, and if so, what are they?

$$Lf = \lambda f \Rightarrow -if'(x) = \lambda f'(x) \Rightarrow f(x) = e^{i\lambda x},\ \lambda \in \mathbb{R}.$$

Problem: $\|e^{i\lambda x}\|_{L_2}^2 = \int_{-\infty}^{\infty} |e^{i\lambda x}|^2\,dx = \int_{-\infty}^{\infty} 1\,dx = +\infty$

$$\Rightarrow e^{i\lambda x} \notin L_2(\mathbb{R};\mathbb{C})!$$

THEREFORE, $e^{i\lambda x}$ CANNOT BE AN EIGENFUNCTION OF L SINCE IT DOES NOT EVEN BELONG TO $L_2(\mathbb{R};\mathbb{C})$!

NOTE: EVEN IF $e^{i\lambda x}$ IS AN E-FUNCTION, NOTE THAT WE WOULD HAVE A CONTINUUM OF E-VALUES $\lambda \in \mathbb{R}$. CONTRAST THIS TO THE FACT THAT FOR $U = [0, a]$ WE HAD A DISCRETE SET OF E-VALUES (CORRESPONDING TO MODES OF THE SYSTEM.

LET'S TRY ANOTHER OPERATOR:

LET $X f(x) \doteq x f(x)$ FOR $f \in L_2(\mathbb{R})$. THEN, X IS SELF-ADJOINT ON $L_2(\mathbb{R})$ SINCE IF $f, g \in L_2(\mathbb{R})$,

$$\langle f | Xg \rangle = \int_{-\infty}^{\infty} f(x)\,(x\, g(x))\, dx = \int_{-\infty}^{\infty} (x f(x))\, g(x)\, dx$$

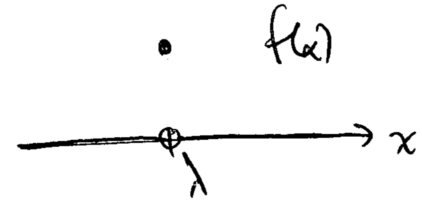
$$= \langle Xf | g \rangle . \checkmark$$

- DOES X HAVE E-FUNCTIONS, AND IF SO, WHAT ARE THEY?

$$Xf = \lambda f \Rightarrow x f(x) = \lambda f(x) \Rightarrow (x - \lambda) f(x) = 0$$

$$\Rightarrow f(x) = 0 \text{ FOR ALL } x \neq \lambda,$$

$\lambda \in \mathbb{R}$.



PROBLEM: $f = 0$ FOR ALL $x \neq \lambda \Rightarrow f \equiv 0$ IN $L_2(\mathbb{R})$ SINCE ALL ELEMENTS WHICH HAVE $\|f\|^2_{L_2} = \int |f|\, dx = 0$ ARE CONSIDERED THE ZERO ELEMENT!

THEREFORE, f CANNOT BE AN E-FUNCTION OF X SINCE E-FUNCTIONS HAVE TO BE NONZERO!

- TO FIND A SOLUTION TO THESE PROBLEMS, WE MUST GENERALIZE OUR NOTION OF E-VALUES / E-VECTORS.

SPECTRUM AND GENERALIZED EIGENFUNCTIONS (9.1):

DEF. THE SPECTRUM $\sigma(A)$ OF AN OPERATOR $A: V \to V$ IS THE SET OF ALL λ S.T. $(A - \lambda I)$ IS NOT CONTINUOUSLY INVERTIBLE. (THAT IS, $(A-\lambda I)^{-1}$ IS NOT A WELL-DEFINED OPERATOR FROM $V \to V$.).

NOTE: λ E-VALUE OF $A \Rightarrow \lambda \in \sigma(A)$

HOWEVER, "$\Leftarrow$" IS NOT TRUE!

EX $V = L_2(\mathbb{R})$, $Xf(x) = xf(x)$.

CONSIDER $f(x) = e^{-x^2} \in L_2(\mathbb{R})$. THEN $0 \in \sigma(X)$ SINCE IF X^{-1} EXISTED, THEN $X^{-1}f(x) = \frac{e^{-x^2}}{x} \notin L_2(\mathbb{R})$.

SO, 0 IS IN THE SPECTRUM OF X EVEN THOUGH IT IS NOT AN E-VALUE, AS WE PREVIOUSLY SAW.

NOTATION: IF $Af = \lambda f$ FOR SOME $\lambda \in \sigma(A)$, WE SAY λ IS A GENERALIZED E-VALUE WITH GENERALIZED E-FUNCTION f. NOTE THAT f ~~MAY~~ NOT BE AN ELEMENT OF V!

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RECALL THE OPERATOR $Xf(x) = xf(x)$. WE SAW LAST TIME THAT $0 \in \sigma(X)$. SIMILARLY, ANY $\lambda \in \mathbb{R}$ IS IN $\sigma(X)$ SINCE IF $(X-\lambda I)^{-1}$ EXISTED THEN WITH $f(x) = e^{-(x-\lambda)^2}$,

$$(X-\lambda I)^{-1} e^{-(x-\lambda)^2} = \frac{e^{-(x-\lambda)^2}}{x-\lambda} \notin L_2(\mathbb{R}).$$

SINCE $\lambda \in \sigma(X)$ HAS NO EIGENFUNCTION IN $L_2(\mathbb{R})$, IT IS <u>NOT</u> AN EIGENVALUE.

<u>Q</u>: CAN WE FIND A GENERALIZED E-FUNCTION ASSOCIATED TO $\lambda \in \sigma(X)$?

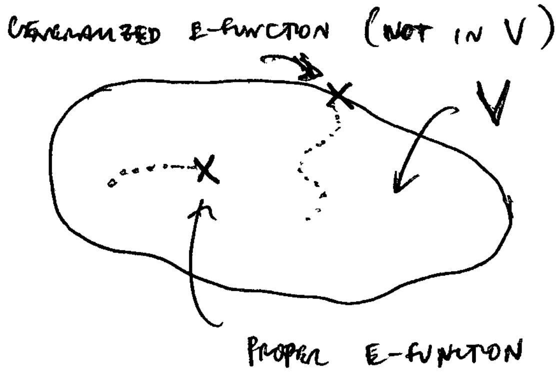
<u>THM.</u> IF $A: V \to V$ IS SELF-ADJOINT,

$\lambda \in \sigma(A) \iff$ THERE IS A SEQUENCE $\{\xi_n\}_{n=1}^{\infty} \subseteq V$ SUCH THAT

$$\left\| (A-\lambda I)\left(\frac{\xi_n}{\|\xi_n\|_V}\right) \right\|_V \xrightarrow{n\to\infty} 0.$$

<u>NOTE:</u> HERE, $\|\cdot\|_V$ IS THE NORM IN V. NOTE THAT WE MAY HAVE $\|\xi_n\|_V \to \infty$ AS $n \to \infty$ SO IT IS POSSIBLE THAT $\xi \doteq \lim_{n\to\infty} \xi_n$ IS NOT AN ELEMENT OF V!

REMARK: FOR A TRUE E-FUNCTION $\xi \in V$ WE HAVE THAT $\left\| (A - \lambda I) \frac{\xi}{\|\xi\|_V} \right\| = 0$. INSTEAD, HERE WE CONSTRUCT GENERALIZED E-FUNCTIONS AS THE LIMIT OF APPROXIMATIONS WHICH ARE IN V (ALTHOUGH THE LIMIT MAY LIE OUTSIDE V!).



• IF $\xi_n \xrightarrow{n \to \infty} \xi \in V$, THEN ξ IS A PROPER E-FUNCTION OF A W/ CORRESPONDING E-VALUE λ.

PF. OF THM.

"$\Rightarrow$" NOT SHOWN.

"$\Leftarrow$" LET $g_n = (A - \lambda I) \frac{\xi_n}{\|\xi_n\|_V}$. THEN $g_n \xrightarrow{n \to \infty} 0$.

BUT IF $(A - \lambda I)$ WERE CONTINUOUSLY INVERTIBLE,

$$\frac{\xi_n}{\|\xi_n\|_V} = (A - \lambda I)^{-1} g_n \to 0 \quad \text{AS } n \to \infty. \quad \text{CONTRADICTION}$$

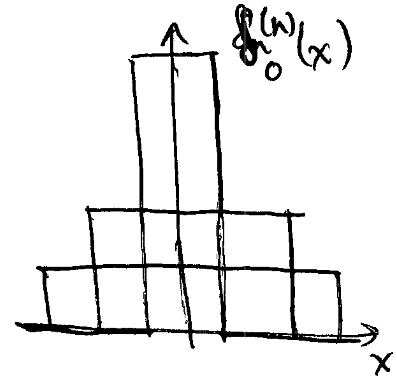
SINCE $\frac{\xi_n}{\|\xi_n\|_V}$ HAS LENGTH 1!

EX. $Xf(x) = xf(x)$, $f \in L_2(\mathbb{R})$.

CONSIDER $\lambda = 0 \in \sigma(X)$. BY THM, WE SHOULD BE ABLE TO CONSTRUCT A SEQUENCE OF APPROXIMATIONS WHICH LIMIT TO A GENERALIZED E-FUNCTION.

LET

$$\delta_0^{(n)}(x) = \begin{cases} 0, & x \notin \left[-\frac{1}{2n}, \frac{1}{2n}\right] \\ n, & x \in \left[-\frac{1}{2n}, \frac{1}{2n}\right] \end{cases}$$



NOTE THAT $\int_{-\infty}^{\infty} \delta_0^{(n)}(x)\,dx = 1$ FOR ALL n,

AND THAT $\delta_0^{(n)}$ BECOMES MORE SHARPLY PEAKED AT $x=0$ AS $n \to \infty$.

WE FIND THAT

$$\|X\delta_0^{(n)}\|_{L_2}^2 = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} |nx|^2\,dx = n^2\left[\frac{x^3}{3}\right]_{-\frac{1}{2n}}^{\frac{1}{2n}} = \frac{1}{12n}$$

$$\|\delta_0^{(n)}\|_{L_2}^2 = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} |n|^2\,dx = n$$

$$\Rightarrow \frac{\|X\delta_0^{(n)}\|_{L_2}^2}{\|\delta_0^{(n)}\|_{L_2}^2} = \left\|(X - 0I)\frac{\delta_0^{(n)}}{\|\delta_0^{(n)}\|_{L_2}}\right\|_{L_2}^2 = \frac{1}{12n^2} \xrightarrow{n\to\infty} 0$$

$\Rightarrow \delta_0 \doteq \lim_{n\to\infty} \delta_0^{(n)}$ IS A GENERALIZED E-FUNCTION OF X W/ GENERALIZED E-VALUE $\lambda = 0$.

<u>Q</u>: WHAT IS $\delta_0 = \lim_{n\to\infty} \delta_0^{(n)}$? SINCE

$\|\delta_0^{(n)}\|_{L_2}^2 = n \xrightarrow{n\to\infty} \infty$ WE SEE THAT THE LIMIT DOES NOT EXIST IN $L_2(\mathbb{R})$. SO, $\delta_0 \notin L_2(\mathbb{R})$, AND IN FACT, ISN'T EVEN A FUNCTION. WHAT IS IT?

<u>DIRAC DELTA DISTRIBUTION (9.2):</u>

- δ_0 IS A <u>DISTRIBUTION</u>, WHICH IS ANY DEFINED THROUGH <u>DUALITY</u> WITH TEST FUNCTIONS $\varphi \in C_0^\infty(\mathbb{R})$ (φ SMOOTH, GOES TO 0 AT $\pm\infty$). THAT IS, FOR ANY SUCH TEST FUNCTION φ,

$$\langle \delta_0 | \varphi \rangle \doteq \int_{-\infty}^{\infty} \delta_0(x)\varphi(x)\,dx$$

$$= \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_0^{(n)}(x)\,\varphi(x)\,dx$$

$$= \lim_{n\to\infty} \left(n \int_{-\frac{1}{2n}}^{\frac{1}{2n}} \varphi(x)\,dx \right)$$

$$= \varphi(0).$$

SINCE $\frac{1}{b-a}\int_a^b \varphi(x)\,dx =$ AVERAGE VALUE OF φ IN $[a,b]$ $\longrightarrow \varphi(c)$ AS $a,b \to c$.

- δ_0 only "sees" the value of φ at $x=0$ when acting on it through the pairing $\langle \delta_0 | \varphi \rangle$.

- A similar argument shows that any $\lambda \in \sigma(X) = \mathbb{R}$ has corresponding generalized e-function

$\delta_\lambda(x) \doteq \delta_0(x-\lambda)$. We have that for any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\langle \delta_\lambda | \varphi \rangle = \int_{-\infty}^{\infty} \delta_\lambda(x)\,\varphi(x)\,dx = \int_{-\infty}^{\infty} \delta_0(x-\lambda)\,\varphi(x)\,dx$$

$$= \int_{-\infty}^{\infty} \delta_0(y)\,\varphi(\lambda+y)\,dy = \varphi(\lambda).$$

Therefore, by self-adjointness of X on $L_2(\mathbb{R})$,

$\{\delta_\lambda(x)\}_{\lambda \in \sigma(X) = \mathbb{R}}$ is an orthogonal basis of generalized e-functions.

$\Rightarrow$ any $f \in L_2(\mathbb{R})$ satisfies

$$\text{"} f = \sum_{\lambda \in \sigma(X)} c_\lambda \delta_\lambda \text{"}$$

that is,

$$f(x) = \int_{\mathbb{R}} c_\lambda \, \delta_\lambda(x)\, d\lambda$$ where

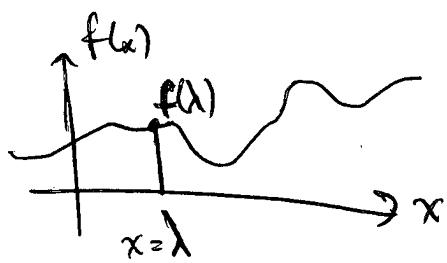
$$c_\lambda = \frac{\langle \delta_\lambda | f \rangle}{\langle \delta_\lambda | \delta_\lambda \rangle} = \int_{-\infty}^{\infty} \delta_\lambda(x)\, f(x)\, dx = f(\lambda)$$

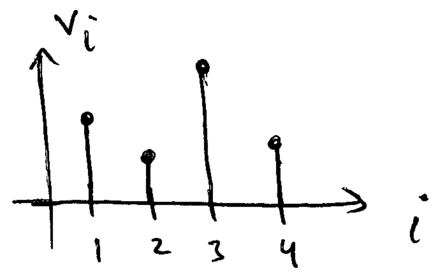
<u>NOTE</u>: FOR TECHNICAL REASONS NOT EXPLAINED HERE, $\langle \delta_\lambda | \delta_\lambda \rangle$ CAN BE TAKEN TO BE 1.

THEREFORE,

$$f(x) = \int_{-\infty}^{\infty} f(\lambda)\, \delta_\lambda(x)\, d\lambda .$$

(THIS IS EXACTLY ANALOGOUS TO WRITING $\underline{v} = \sum_{i=1}^{N} v_i \underline{e}_i$ WITH $\{\underline{e}_i\}_{i=1}^{N}$ THE STANDARD BASIS IN $\mathbb{R}^N$!





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<u>REMARK</u>: WE CAN CHECK THE ABOVE EQUATION DIRECTLY AS FOLLOWS. USING THAT $\delta_0(x-\lambda) = \delta_0(\lambda - x)$ BY SYMMETRY,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\lambda)\, \delta_\lambda(x)\, d\lambda &= \int_{-\infty}^{\infty} f(\lambda)\, \delta_0(x-\lambda)\, d\lambda \\
&= \int_{-\infty}^{\infty} f(\lambda)\, \delta_0(\lambda - x)\, d\lambda \\
&= \int_{-\infty}^{\infty} f(\lambda)\, \delta_x(\lambda)\, d\lambda \\
&= f(x). \checkmark
\end{aligned}$$

WE CAN ALSO VERIFY DIRECTLY THAT δ_λ IS A GENERALIZED E-FUNCTION OF X W/ GENERALIZED E-VALUE λ:

LET $\varphi \in C_0^\infty(\mathbb{R})$ BE ANY TEST FUNCTION. THEN,

$$\langle X\delta_\lambda | \varphi \rangle = \langle \delta_\lambda | X\varphi \rangle$$ (BY SELF-ADJOINTNESS OF X)

$$= \int_{-\infty}^{\infty} \delta_\lambda(x)\, x\varphi(x)\, dx$$

$$= \lambda \varphi(\lambda)$$

$$\langle \lambda \delta_\lambda | \varphi \rangle = \int_{-\infty}^{\infty} \lambda\, \delta_\lambda(x) \varphi(x)\, dx$$

$$= \lambda \varphi(\lambda)$$

$\Rightarrow \langle X\delta_\lambda | \varphi \rangle = \langle \lambda \delta_\lambda | \varphi \rangle$ FOR ALL $\varphi \in C_0^\infty(\mathbb{R})$

$$\Rightarrow X\delta_\lambda = \lambda \delta_\lambda. \quad ✓$$

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LAST TIME WE FOUND THE DECOMPOSITION

$$f(x) = \int_{-\infty}^{\infty} f(\lambda)\, \delta_\lambda(x)\, d\lambda \quad , \quad f \in L_2(\mathbb{R}).$$

USING THE BASIS OF ORTHOGONAL GENERALIZED E-FUNCTIONS OF THE SELF-ADJOINT OPERATOR $X f(x) = x f(x)$.

Q: CAN WE DO SOMETHING SIMILAR FOR THE SELF-ADJOINT OPERATOR $L = -i \frac{d}{dx}$ ON $L_2(\mathbb{R};\mathbb{C})$?

WE HAVE ALREADY SEEN THAT

$$L e^{i\lambda x} = \lambda e^{i\lambda x} \quad , \quad \lambda \in \sigma(L) = \mathbb{R},$$

BUT $e^{i\lambda x} \notin L_2(\mathbb{R};\mathbb{C})$, SO THESE ARE NOT PROPER E-FUNCTIONS BUT RATHER GENERALIZED E-FUNCTIONS. TO CONFIRM THIS, LET US CONSTRUCT A SEQUENCE OF APPROXIMATIONS $\{\xi_n\}_{n=1}^{\infty}$ SUCH THAT

$$\left\| (L - \lambda I) \frac{\xi_n}{\|\xi_n\|_{L_2}} \right\|_{L_2}^2 \xrightarrow{n\to\infty} 0 .$$

LET $\xi_n(x) := e^{i\lambda x}\,\eta_n(x)$ FOR SOME η_n TO BE DETERMINED.

THEN,

$$(L-\lambda I)\,\xi_n = -i\,\frac{d}{dx}\left(e^{i\lambda x}\,\eta_n(x)\right) - \lambda e^{i\lambda x}\,\eta_n(x)$$

$$= \cancel{\lambda e^{i\lambda x}\,\eta_n(x)} - i e^{i\lambda x}\,\eta_n'(x) - \cancel{\lambda e^{i\lambda x}\,\eta_n(x)}$$

$$= -i e^{i\lambda x}\,\eta_n'(x).$$

$$\Rightarrow \quad \|(L-\lambda I)\,\xi_n\|_{L_2}^2 = |-i|^2 \int_{-\infty}^{\infty} |e^{i\lambda x}|^2\,|\eta_n'(x)|^2\,dx$$

$$= \|\eta_n'\|_{L_2}^2.$$

$$\|\xi_n\|_{L_2}^2 = \int_{-\infty}^{\infty} |e^{i\lambda x}|^2\,|\eta_n(x)|^2\,dx = \|\eta_n\|_{L_2}^2$$

$$\Rightarrow \quad \left\|(L-\lambda I)\,\frac{\xi_n}{\|\xi_n\|_{L_2}}\right\|_{L_2}^2 = \frac{\|\eta_n'\|_{L_2}^2}{\|\eta_n\|_{L_2}^2} \xrightarrow{n\to 0} 0$$

IF WE TAKE $\eta_n(x) = e^{-x^2/n^2}$ SINCE

$$\|\eta_n'\|_{L_2}^2 = \int_{-\infty}^{\infty} \frac{4x^2}{n^4}\,e^{-2x^2/n^2}\,dx = \frac{1}{n}\int_{-\infty}^{\infty} 4y^2 e^{-2y^2}\,dy$$

$$= \text{CONST.} \times \frac{1}{n}$$

AND $\|\eta_n\|^2_{L_2} = \int_{-\infty}^{\infty} e^{-2x^2/n^2}\,dx = n\int_{-\infty}^{\infty} e^{-2y^2}\,dy$

$= \text{CONST.} \times n$

IMPLIES $\dfrac{\|\eta_n'\|^2_{L_2}}{\|\eta_n\|^2_{L_2}} = \text{CONST.} \times \dfrac{1}{n^2} \xrightarrow{n\to\infty} 0$.

THEREFORE, SINCE $\eta_n(x) = e^{-x^2/n^2} \xrightarrow{n\to\infty} 1$,

$\xi_n(x) = e^{i\lambda x}\eta_n(x) \longrightarrow e^{i\lambda x} \doteq \xi$ IS

A GENERALIZED E-FUNCTION ASSOCIATED TO $\lambda \in \sigma(L)$.

TO SUMMARIZE,

$\{e^{i\lambda x}\}_{\lambda\in\sigma(L)=\mathbb{R}}$ IS AN ORTHOGONAL BASIS OF GENERALIZED E-FUNCTIONS.

$\Rightarrow$ ANY $f \in L_2(\mathbb{R})$ SATISFIES

"$f(x) = \sum_{\lambda\in\sigma(L)} c_\lambda e^{i\lambda x}$"

THAT IS,

$f(x) = \int_{\mathbb{R}} c_\lambda e^{i\lambda x}\,d\lambda$ WHERE

$$c_\lambda = \frac{\langle e^{i\lambda x} | f \rangle}{\langle e^{i\lambda x} | e^{i\lambda x} \rangle} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{e^{i\lambda x}} f(x)\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx .$$

<u>NOTE:</u> FOR TECHNICAL REASONS, WE CAN REPLACE $\langle e^{i\lambda x} | e^{i\lambda x} \rangle$ BY 2π.

<u>NOTATION:</u> IN MOST TEXTS, $\sqrt{2\pi}\, c_\lambda$ IS DENOTED $\hat{f}(\lambda)$.

THEN,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda)\, e^{i\lambda x}\, d\lambda$$

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx .$$

$\hat{f}(\lambda)$ IS CALLED THE <u>FOURIER TRANSFORM</u> OF f AND TELLS US "HOW MUCH" OF THE COMPONENT $e^{i\lambda x}$ IS PRESENT IN OUR FUNCTION $f \in L_2(\mathbb{R})$.

<u>EX.</u> $f(x) = e^{-x^2/2}$ (GAUSSIAN).

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} - ikx}\, dx$$

$$= e^{-\frac{\lambda^2}{2}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x+i\lambda)^2}{2}} dx \right)$$

(BY USING THAT $(x+i\lambda)^2 = x^2 + 2i\lambda x - \lambda^2$ AND COMPLETING THE SQUARE IN THE EXPONENT.)

$$= e^{-\frac{\lambda^2}{2}} \underbrace{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \right)}_{=1} \quad \text{(BY LETTING } y = x+i\lambda\text{)}$$

($\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$ IS THE PROBABILITY DENSITY FUNCTION OF A NORMAL RANDOM VARIABLE, SO IT INTEGRATES TO 1).

$$= e^{-\frac{\lambda^2}{2}}.$$

SO, THE GAUSSIAN FUNCTION HAS THE SPECIAL PROPERTY OF BEING EQUAL TO ITS FOURIER TRANSFORM!

(<u>REMARK</u>: TECHNICALLY, WE HAD TO USE A CONTOUR DEFORMATION FOR THE INTEGRAL ABOVE:

$$\int_{-\infty}^{\infty} e^{-\frac{(x+i\lambda)^2}{2}} dx = \int_{-\infty+i\lambda}^{\infty+i\lambda} e^{-\frac{y^2}{2}} dy = \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$

SINCE $e^{-\frac{y^2}{2}}$ IS <u>HOLOMORPHIC</u> IN $\mathbb{C}$.)

- TO CONCLUDE, LET US NOTE THAT MANY OF THE PROPERTIES THAT WE FOUND FOR FOURIER SERIES HOLD FOR THE FOURIER TRANSFORM AS WELL.

(i) DECAY RATE OF $\hat{f}(\lambda)$ AS $|\lambda| \to \infty$ CORRESPONDS TO SMOOTHNESS OF $f(x)$.

(ii) FOURIER TRANSFORM CONVENIENT TO USE WHEN WORKING W/ DERIVATIVES. SINCE

$$L e^{i\lambda x} = -i \frac{d}{dx} e^{i\lambda x} = \lambda e^{i\lambda x},$$

$$L f(x) = -i \frac{d}{dx} f(x) = \int_{-\infty}^{\infty} \hat{f}(\lambda)\, \lambda\, e^{i\lambda x}$$ AND

FOURIER TRANSFORM OF Lf IS $\lambda \hat{f}(\lambda)$ (I.E., DERIVATIVES IN REAL SPACE CORRESPOND TO MULTIPLICATION BY $i\lambda$ IN FOURIER SPACE!

THIS IS VERY USEFUL WHEN WORKING WITH PDE. FOR EX,

$$\begin{cases} \partial_t u = \partial_{xx} u, & x \in \mathbb{R} \\ u(x,0) \text{ given} & t \geq 0 \end{cases} \xrightarrow{\text{Fourier transform}} \begin{cases} \frac{d}{dt}\hat{u} = -\lambda^2 \hat{u} \\ \hat{u}(\lambda, 0) \text{ given}, \end{cases}$$

$\lambda \in \mathbb{R},\ t \geq 0.$

$\Rightarrow \hat{u}(\lambda, t) = \hat{u}(\lambda, 0)\, e^{-\lambda^2 t}$. NOW FOURIER TRANSFORM BACK TO x.