LECTURE 12
02/15/12

COMPLEXIFICATION (4.4):

$A \in M_{n,n}(\mathbb{R})$

$p_A(z) = \det(A - zI)$ DEGREE n POLYNOMIAL IN z (USE COFACTOR EXPANSION TO SEE THIS)

PROBLEM: MAY NOT HAVE ENOUGH REAL ROOTS OF $p_A(z)$ TO FIND n EIGENVALUES OF A! TO REMEDY THIS, LOOK FOR COMPLEX ROOTS OF p_A.

THM. (FUNDAMENTAL THM. OF ALGEBRA)
ANY n DEGREE POLYNOMIAL (WITH COMPLEX COEFF.) HAS n (POSSIBLY REPEATED) ROOTS IN $\mathbb{C}$.

$$\Rightarrow \quad p_A(z) = c_0 + c_1 z + \dots + c_n z^n \qquad \left(\begin{array}{lll} c_i \in \mathbb{R} & \text{IF} & A \in M_{n,n}(\mathbb{R}) \\ c_i \in \mathbb{C} & \text{IF} & A \in M_{n,n}(\mathbb{C}) \end{array}\right).$$
$$= c_n (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$$

WITH $\lambda_i \in \mathbb{C}$ POSSIBLY REPEATED. THESE ARE THE EIGENVALUES OF A. WHAT ARE THE CORRESPONDING EIGENVECTORS?

BY ALLOWING FOR COMPLEX EIGENVALUES, WE HAVE IMPLICITY MOVED FROM CONSIDERING THE ACTION OF THE OPERATOR $A \in M_{n,n}(\mathbb{R})$ ON THE REAL VECTOR SPACE $\mathbb{R}^n$ TO THE COMPLEX VECTOR SPACE $\mathbb{C}^n$. THIS IS KNOWN AS COMPLEXIFICATION.

REAL VECTOR SPACE $\xrightarrow{\text{COMPLEXIFICATION}}$ COMPLEX VECTOR SPACE

V $\longrightarrow$ $V_c \doteq \{\underline{v} + i\underline{w} : \underline{v}, \underline{w} \in V\}$

SCALARS $c \in \mathbb{R}$ | SCALARS $c \in \mathbb{C}$.

BASIS $\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ | BASIS $\mathcal{B}_c = \{\underline{b}_1 + i\underline{0}, \ldots, \underline{b}_n + i\underline{0}\}$.

$V \simeq \mathbb{R}^n$

(I.E, V ISOMORPHIC TO n-DIM REAL SPACE)

$V_c \simeq \mathbb{C}^n$

(I.E, V_c ISOMORPHIC TO n-DIM. COMPLEX SPACE)

NOTE: $\mathbb{C}^n \simeq \mathbb{R}^{2n}$

(I.E, $\mathbb{C}^n$ ISOMORPHIC TO n-DIM REAL SPACE).

$L : V \to V$ LINEAR $\longrightarrow$ $L : V_c \to V_c$ GIVEN BY

$$L(\underline{v} + i\underline{w}) \doteq L(\underline{v}) + i\, L(\underline{w}).$$

WITH THIS, WE CAN CONSIDER COMPLEX EIGENVALUES / EIGENVECTORS OF ANY $A \in M_{n,n}(\mathbb{C})$.

DEF. $\lambda \in \mathbb{C}$ EIGENVALUE W/ CORRESPONDING EIGENVECTOR $\underline{\xi} \in \mathbb{C}^n$ IF

$$A\underline{\xi} = \lambda \underline{\xi} \quad (\text{AND } \underline{\xi} \neq \underline{0}).$$

THM. IF $A \in M_{n,n}(\mathbb{R})$ (I.E, IF A IS A REAL MATRIX) COMPLEX EIGENVALUES / EIGENVECTORS ALWAYS COME IN COMPLEX CONJUGATE PAIRS. THAT IS,

$$\underbrace{\lambda = a + ib}_{\text{E-VALUE}} \text{ w/ } \underbrace{\underline{\xi} = \underline{v} + i\underline{w}}_{\text{E-VECTOR}} \iff \underbrace{\overline{\lambda} = a - ib}_{\text{E-VALUE}} \text{ w/ } \underbrace{\overline{\underline{\xi}} = \underline{v} - i\underline{w}}_{\text{E-VECTOR}}.$$

PF. $A\underline{\xi} = \lambda \underline{\xi} \iff \overline{A\underline{\xi}} = \overline{\lambda \underline{\xi}}$

$$\iff A\overline{\underline{\xi}} = \overline{\lambda}\,\overline{\underline{\xi}}$$

SINCE $\overline{A} = A$ (ALL ENTRIES OF A ARE REAL).

REMARK: IF A IS AN $n \times n$ REAL MATRIX AND $n \in \mathbb{N}$ IS ODD, A MUST HAVE AT LEAST ONE REAL EIGENVALUE!

WE CAN NOW EASILY FIND n EIGENVALUES AND THEIR CORRESPONDING EIGENSPACES AS BEFORE, EXCEPT WE CAN NOW USE MULTIPLICATION WITH COMPLEX-VALUED SCALARS WHEN PERFORMING ANY ROW REDUCTION OPERATIONS.

NOTE: IF $z = a + ib \in \mathbb{C}$, WHERE $a = \mathrm{Re}(z) \in \mathbb{R}$ AND $b = \mathrm{Im}(z) \in \mathbb{R}$ ARE THE REAL AND IMAGINARY PARTS OF z, THE MAGNITUDE OF z IS DEFINED BY

$$|z|^2 = z\overline{z} = (a+ib)(a-ib) = a^2+b^2.$$

THIS IS USEFUL WHEN PERFORMING ROW OPERATIONS. INSTEAD OF MULTIPLYING BY $\frac{1}{z}$, SIMPLY MULTIPLY BY $\frac{\overline{z}}{|z|^2}$ IN ORDER TO GET THE SAME RESULT. FOR EXAMPLE,

$$\frac{1}{3+4i} = \frac{1}{3+4i}\cdot\frac{3-4i}{3-4i} = \frac{3-4i}{3^2+4^2} = \frac{1}{25}(3-4i).$$

Ex. FIND EIGENVALUES AND CORRESPONDING EIGENSPACES OF

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}.$$

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & -2 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (3-\lambda)\Big[\underbrace{(1-\lambda)^2 + 2^2}\Big] = 0$$

IF THIS IS ZERO, THEN

$$(1-\lambda)^2 = -2^2$$

$$\Rightarrow 1-\lambda = \pm\sqrt{-2^2} = \pm 2i$$

$$\Rightarrow \lambda = 1 \pm 2i.$$

SO, THE EIGENVALUES OF A ARE $\lambda = 3$, $\underbrace{\lambda = 1+2i\ ,\ \lambda = 1-2i}_{\text{COMPLEX CONJUGATE PAIR.}}$

$\underline{\underline{E_3}}$:

$$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -2 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \underline{x} = x_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \{(1,0,0)^T\} \text{ BASIS OF } E_3.$$

$\underline{\underline{E_{1+2i}}}$:

$$A - (1+2i)I = \begin{pmatrix} 2-2i & 0 & 0 \\ 0 & -2i & 2 \\ 0 & -2 & -2i \end{pmatrix}\begin{matrix} \times \frac{2+2i}{8} \\ \times \frac{2i}{4} \\ \times -\frac{1}{2} \end{matrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & 1 & i \end{pmatrix}_{-R_2} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & i \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} = -i\xi_3 \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}.$$

$$\Rightarrow \quad \{(0,1,i)^T\} \quad \text{BASIS FOR} \quad E_{1+2i}.$$

$\underline{\underline{E_{1-2i}}}$: NO NEED TO DO EXTRA WORK! SINCE COMPLEX EIGENVALUES / EIGENVECTORS OF <u>REAL</u> MATRICES COME IN PAIRS, WE HAVE THAT $\overline{(0,1,i)^T} = (0,1,-i)^T$, SO

$$\Rightarrow \quad \{(0,1,-i)^T\} \quad \text{BASIS FOR} \quad E_{1-2i}.$$

LECTURE 13
02/17/12

Q: WHAT IS THE MEANING OF COMPLEX EIGENVALUE/EIGENVECTORS PAIRS OBTAINED FOR A <u>REAL</u> MATRIX A?

A: <u>ROTATION</u> AND <u>SCALING</u> IN THE PLANE SPANNED BY THE REAL AND IMAGINARY PARTS OF THE COMPLEX EIGENVECTOR PAIR.

TO SEE THIS, SUPPOSE $\lambda = a + ib$, $b \neq 0$, IS AN EIGEN~~VALUE~~ OF A. THEN SO IS $\bar{\lambda} = a - ib$.
THEIR CORRESPONDING EIGENVECTORS ARE $\underline{\xi} = \underline{v} + i\underline{w}$ AND $\bar{\underline{\xi}} = \underline{v} - i\underline{w}$ (NOTE THAT SINCE $b \neq 0$, $\underline{w} \neq \underline{0}$).

HOW DOES A ACT ON THE REAL VECTORS $\underline{v}$ AND $\underline{w}$?

$$\underline{v} = \frac{\underline{\xi} + \bar{\underline{\xi}}}{2} \Rightarrow A\underline{v} = A\left(\frac{\underline{\xi} + \bar{\underline{\xi}}}{2}\right) = \frac{\lambda\underline{\xi} + \bar{\lambda}\bar{\underline{\xi}}}{2}$$

$$= \left(\frac{\lambda + \bar{\lambda}}{2}\right)\underline{v} + i\left(\frac{\lambda - \bar{\lambda}}{2}\right)\underline{w}$$

$$= a\underline{v} - b\underline{w}.$$

$$\underline{w} = \frac{\underline{\xi} - \bar{\underline{\xi}}}{2i} \Rightarrow A\underline{w} = b\underline{v} + a\underline{w}$$

BY SIMILAR COMPUTATION.

ASSUMING $\underline{z}$ AND $\overline{\underline{z}}$ ARE LINEARLY INDEPENDENT SINCE THEY CORRESPOND TO DIFFERENT EIGENVALUES (WE WILL PROVE THIS LATER), $\underline{v}$ AND $\underline{w}$ ARE LINEARLY INDEPENDENT AND SPAN A PLANE

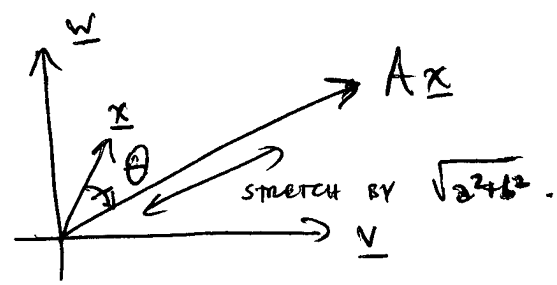
$$P = \text{span}\{\underline{v}, \underline{w}\} \quad \text{in } \mathbb{R}^n .$$

WE HAVE JUST SHOWN THAT $A : P \to P$, AND IS REPRESENTED IN THE BASIS $\mathcal{D} = \{\underline{v}, \underline{w}\}$ OF P BY

$$[A]_{\mathcal{D}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \longleftarrow \text{THIS IS A ROTATION MATRIX IN } \mathbb{R}^2 !$$

$$\begin{matrix} \| & \| \\ [A\underline{v}]_{\mathcal{D}} & [A\underline{w}]_{\mathcal{D}} \end{matrix}$$

THAT IS, IN PLANE P, A ROTATES VECTORS BY ANGLE $\theta = \arctan\left(-\frac{b}{a}\right)$ AND STRETCHES THEM BY A FACTOR OF $|\lambda| = \sqrt{a^2+b^2}$:



$\underline{\text{Ex.}}$ IN PREVIOUS LECTURE, WE FOUND THAT

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

HAS EIGENVALUES / EIGENSPACES GIVEN BY

$$\lambda = 3, \quad \lambda = 1+2i, \quad \lambda = 1-2i$$

$$\{(1,0,0)^T\} \quad \underbrace{\{(0,1,i)^T\} \quad \{(0,1,-i)^T\}}_{\text{COMPLEX CONJUGATE PAIR.}}$$

- THE REAL EIGENVALUE 3 w/ EIGENVECTOR $(1,0,0)^T$ IMPLIES THAT THE ACTION OF A STRETCHES ALL VECTORS IN $\mathbb{R}^3$ IN THE X-DIRECTION BY A FACTOR OF 3.

- $\lambda = 1 \pm 2i$ w/ CORRESPONDING EIGENVECTORS

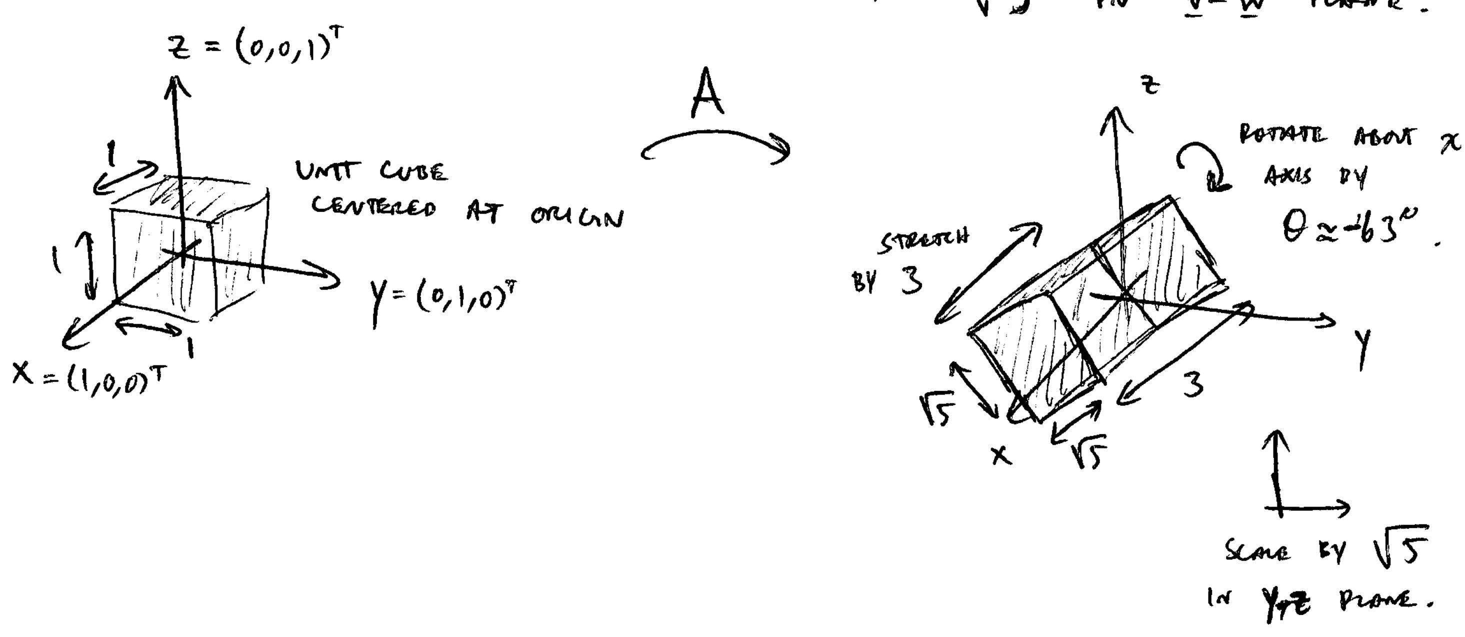
$$\underline{\xi} = (0,1,\pm i)^T \implies a=1,\ b=2$$

$$= (0,1,0)^T \pm i(0,0,1)^T \qquad \underline{v} = (0,1,0)^T,\ \underline{w} = (0,0,1)^T.$$

$$\implies \theta = \arctan\left(-\frac{b}{a}\right) = \arctan(-2) \approx -63^\circ.$$

$$|\lambda| = \sqrt{a^2+b^2} = \sqrt{5}.$$

$\implies$ ROTATE BY $\theta \approx -63^\circ$ IN $\underline{v}$-$\underline{w}$ PLANE (I.E., y-z PLANE) AND SCALE BY $\sqrt{5}$ IN $\underline{v}$-$\underline{w}$ PLANE.



BY ALLOWING FOR COMPLEX EIGENVALUES, WE HAVE NOW GIVEN OURSELVES A CHANCE OF FINDING ENOUGH EIGENVECTORS TO FORM A BASIS OF $\mathbb{R}^n$ (IN ORDER TO DIAGONALIZE A). BUT WHAT IF WE STILL CAN'T FIND ENOUGH LINEARLY INDEP. EIGENVECTORS?

MULTIPLICITY OF EIGENVALUES, DIAGONALIZABILITY (4.5):

$$p_A(z) = c_n (z-\tilde{\lambda}_1)(z-\tilde{\lambda}_2)\cdots(z-\tilde{\lambda}_n), \quad \tilde{\lambda}_i \text{ POSSIBLY REPEATED}$$

$$= c_n (z-\lambda_1)^{m_1}(z-\lambda_2)^{m_2}\cdots(z-\lambda_r')^{m_r}, \quad \lambda_1,\dots,\lambda_r \in \mathbb{C} \text{ } \underline{\text{NOT}} \text{ REPEATED.}$$

NOTE THAT $m_1 + \cdots + m_r = n$.

EX. IF $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

$$p_A(z) = (\lambda - 1)(\lambda - 1) \qquad (\text{I.E., } \tilde{\lambda}_1 = 1,\ \tilde{\lambda}_2 = 1)$$

$$= (\lambda - 1)^2 \qquad (\text{I.E., } \lambda_1 = 1, \text{ WITH } m_1 = 2).$$

DEF
- $m_i \geq 1$, $i = 1,\dots,r$, ARE THE <u>ALGEBRAIC MULTIPLICITIES</u> OF THE EIGENVALUES λ_i, $i=1,\dots,r$.
- $M_i \doteq \dim(\mathrm{Ker}(A - \lambda_i I)) \geq 1$, $i=1,\dots,r$, ARE THE <u>GEOMETRIC MULTIPLICITIES</u> OF THE λ_i, $i=1,\dots,r$.

NOTE: IN SADUN, m_i IS DENOTED $m_a(\lambda_i)$ AND M_i IS DENOTED $m_g(\lambda_i)$.

NOTE THAT M_i IS SIMPLY THE DIMENSION OF THE EIGENSPACE E_{λ_i} AND TELLS US HOW MANY LINEARLY INDEP. EIGENVECTORS WE CAN GET FROM EIGENVALUE λ_i.

<u>THM.</u> $1 \leq M_i \leq m_i$ FOR ALL $i = 1, \dots, r$

(I.E., GEOMETRIC MULTIPLICITY $\leq$ ALGEBRAIC MULTIPLICITY).

<u>PF.</u> LET λ BE AN EIGENVALUE OF $A \in M_{n,n}$ W/ ALG. MULT. m AND GEOM. MULT. M. SUPPOSE E_λ HAS BASIS $\{\underline{b}_1, \dots, \underline{b}_M\}$.

THEN, WE CAN FIND A BASIS OF $\mathbb{R}^n$ GIVEN BY

$$\mathcal{B} = \{\underbrace{\underline{b}_1, \dots, \underline{b}_M}_{\text{IN } E_\lambda}, \underline{b}_{M+1}, \dots, \underline{b}_n\}.$$

WHAT IS $[A]_{\mathcal{B}}$?

$$[A]_{\mathcal{B}} = \left([A\underline{b}_1]_{\mathcal{B}} \cdots [A\underline{b}_M]_{\mathcal{B}} \; [A\underline{b}_{M+1}]_{\mathcal{B}} \cdots [A\underline{b}_n]_{\mathcal{B}} \right)$$

$$= \left(\lambda\underline{e}_1 \cdots \lambda\underline{e}_M \; [A\underline{b}_{M+1}]_{\mathcal{B}} \cdots [A\underline{b}_n]_{\mathcal{B}} \right)$$

$$= \begin{array}{c c} & \begin{array}{cc} \xleftrightarrow{M} & \xleftrightarrow{n-M} \end{array} \\ \begin{array}{c} M \updownarrow \\ n-M \updownarrow \end{array} & \left(\begin{array}{c|c} \lambda I & \tilde{B} \\ \hline 0 & B \end{array} \right) \end{array}$$

FOR SOME MATRICES

$B \in M_{n-M, n-M}$
$\tilde{B} \in M_{M, n-M}$.

Therefore,

$$p_A(z) = p_{[A]_\beta}(z) = \det\left([A]_\beta - zI \right)$$

E-values indep. of basis.

$$= \left| \begin{array}{ccc|c} \lambda - z & & & \\ & \ddots & & \tilde{B} \\ & & \lambda - z & \\ \hline & 0 & & B - zI \end{array} \right|$$

$$= \underbrace{(\lambda - z)^M}_{(-1)^M (z-\lambda)^M} \; \underbrace{\det(B - zI)}_{n-M \text{ degree polynomial}}$$

$\Rightarrow$ m has to be at least M!
(since other factors of $(z-\lambda)$ may be in $\det(B - zI)$.) ∎

Furthermore, we have:

<u>Thm.</u> If $\lambda_1, \ldots, \lambda_r$ are distinct eigenvalues w/ corresponding eigenvectors $\underline{\xi}_1, \ldots, \underline{\xi}_r$, then $\{\underline{\xi}_1, \ldots, \underline{\xi}_r\}$ are linearly independent.

<u>Pf.</u> (By induction).

Suppose $\{\underline{\xi}_1, \ldots, \underline{\xi}_{k-1}\}$ lin. indep. for some $k \leq r$.

Then $a_1 \underline{\xi}_1 + \cdots + a_{k-1} \underline{\xi}_{k-1} = \underline{0} \Rightarrow a_1 = \cdots = a_{k-1} = 0$.

NOW SUPPOSE $c_1 \underline{\xi}_1 + \cdots + c_{k-1}\underline{\xi}_{k-1} + c_k \underline{\xi}_k = \underline{0}$

FOR SOME $\{c_i\}_{i=1}^{k}$. WE WOULD LIKE TO SHOW

THAT $c_i = 0$ FOR ALL i. TO DO THIS, APPLY

$A - \lambda_k I$ TO BOTH SIDES TO GET

$$\begin{aligned}\underline{0} &= (A-\lambda_k I)(c_1\underline{\xi}_1 + \cdots + c_{k-1}\underline{\xi}_{k-1} + c_k\underline{\xi}_k) \\ &= c_1(\lambda_1-\lambda_k)\xi_1 + \cdots + c_{k-1}(\lambda_1-\lambda_k)\xi_{k-1} + \underline{0}. \\ &= \sum_{i=1}^{k-1} \underbrace{c_i(\lambda_i-\lambda_k)}_{=0 \text{ BY LINEAR INDEP. OF } \xi_1,\ldots,\xi_{k-1}} \xi_i .\end{aligned}$$

$\Rightarrow \quad c_i(\lambda_i - \lambda_k) = 0$ FOR $i = 1, \ldots, k-1$

$\Rightarrow \quad c_i = 0$ FOR $i=1,\ldots,k-1$ (SINCE $\lambda_i \neq \lambda_k$)

$\Rightarrow \quad c_k = 0$ (SINCE $c_k \underline{\xi}_k = \underline{0} \Rightarrow c_k = 0$).

SO, $\{\xi_i\}_{i=1}^{k}$ ARE LINEARLY INDEP.

SINCE TRIVIALLY TRUE FOR $k=1$, WE ARE DONE.

∎

<u>COROLLARY</u>: IF $\lambda_1, \ldots, \lambda_r$ ARE <u>DISTINCT</u> EIGENVALUES, THEIR CORRESPONDING EIGENSPACES $E_{\lambda_1}, \ldots, E_{\lambda_r}$ ARE LINEARLY INDEP.

<u>PF.</u>

$$\underline{0} = \underbrace{\left(a_1^{(1)}\underline{b}_1^{(1)} + \cdots + a_{m_1}^{(1)}\underline{b}_{m_1}^{(1)}\right)}_{\doteq d_1 \in E_{\lambda_1}} + \cdots + \underbrace{\left(a_1^{(r)}\underline{b}_1^{(r)} + \cdots + a_{m_r}^{(r)}\underline{b}_{m_r}^{(r)}\right)}_{\doteq d_r \in E_{\lambda_r}}$$

$\Rightarrow \underline{0} = \underline{d}_1 + \cdots + \underline{d}_r \Rightarrow \underline{d}_i = \underline{0}$ FOR ALL $i = 1,\ldots,r$ (BY PREVIOUS THM.)

$\Rightarrow a_1^{(i)} = \cdots = a_{m_i}^{(i)} = 0$ FOR ALL $i = 1, \ldots, r$.

∎