LECTURE 14
02/20/12

MULTIPLICITY OF EIGENVALUES, DIAGONALIZABILITY (CONT'D):

TO RECAP: $A \in M_{n,n}$ HAS <u>DISTINCT</u> EIGENVALUES $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$.

$$(\lambda_1, E_{\lambda_1}), \quad \ldots \quad, (\lambda_r, E_{\lambda_r})$$

$$(z-\lambda_1)^{m_1} \downarrow \quad \downarrow \dim E_{\lambda_1} \qquad (z-\lambda_r)^{m_r} \downarrow \quad \downarrow \dim E_{\lambda_r}$$

$$m_1 \quad M_1 \qquad\qquad m_r \quad M_r$$

- m_i ALGEBRAIC MULTIPLICITY OF λ_i
- M_i GEOMETRIC MULTIPLICITY OF λ_i.

LAST TIME, WE SHOWED THAT

(i) $1 \leq M_i \leq m_i$ FOR ALL $i = 1, \ldots, r$

(ii) $E_{\lambda_1}, \ldots, E_{\lambda_r}$ ARE <u>DISTINCT</u> SUBSPACES IN THAT THEY ARE LINEARLY INDEP.

THESE IMMEDIATELY IMPLY:

<u>THM.</u> (DIAGONALIZABILITY)

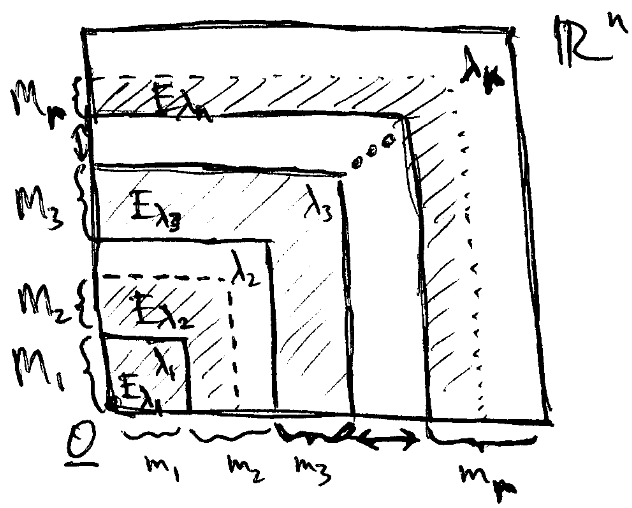
$A \in M_{n,n}$ DIAGONALIZABLE, I.E, CAN FIND n LINEARLY INDEP. EIGENVECTORS

$(A = PDP^{-1}$ FOR SOME D DIAGONAL$)$

$\iff M_i = m_i$ FOR ALL $i = 1, \ldots, r$.

• In particular, if A has n distinct eigenvalues (i.e., $r = n$ and $m_1 = m_2 = \cdots = m_n = 1$) then $M_1 = M_2 = \cdots = 1$ and A is diagonalizable.

To remember all of this in a picture, imagine that each distinct eigenvalue λ_i "reserves" a part of $\mathbb{R}^n$ for itself, and its eigenspace E_{λ_i} must sit inside this reserved space (and takes up all of it if the geometric mult. of λ_i matches its algebraic mult.). That is,



$\lambda_1, \ldots, \lambda_r \in \mathbb{C}$

distinct eigenvalues.

A diagonalizable $\iff M_i = m_i$ for all i.

<u>Q:</u> What if $M_i < m_i$ for some i?

<u>A:</u> A is not diagonalizable, but is <u>almost</u> diagonalizable.

Jordan canonical form (4.9):

If A is diagonalizable, then $A = PDP^{-1}$, where

$$D = \underbrace{\left.\begin{pmatrix} \overbrace{\lambda_1 \ \cdots \ \lambda_1}^{M_1 = m_1} & & & \\ & \overbrace{\lambda_2 \ \cdots \ \lambda_2}^{M_2 = m_2} & & \\ & & \ddots & \\ & & & \overbrace{\lambda_r \ \cdots \ \lambda_r}^{M_r = m_r} \end{pmatrix}\right\updownarrow n}_{n},$$

$$P = \left(\begin{array}{c|c|c|c} \overbrace{\underbrace{\text{Eigenvectors of } \lambda_1}_{M_1}}^{E_{\lambda_1}} & \overbrace{\underbrace{\text{Eigenvectors of } \lambda_2}_{M_2}}^{E_{\lambda_2}} & \cdots & \overbrace{\underbrace{\text{Eigenvectors of } \lambda_r}_{M_r}}^{E_{\lambda_r}} \end{array} \right).$$

<u>Q</u>: If $M_i \neq m_i$ for all i, we are "missing" eigenvectors. Can we substitute these with a more general notion of eigenvector?

<u>A</u>: Yes. This leads to <u>generalized eigenvectors</u> (<u>power vectors</u>).

Suppose eigenvalue λ of A has alg. mult. m, geom. mult. M.

- $E_\lambda = \operatorname{Ker}(A - \lambda I)$ has basis $\mathcal{B}_\lambda = \{\underline{b}_1, \dots, \underline{b}_M\}$ and dimension $M \leq m$.
  ↳ eigenspace associated to λ
- $\tilde{E}_\lambda = \operatorname{Ker}((A - \lambda I)^m)$ has basis $\tilde{\mathcal{B}}_\lambda = \{\underline{b}_1, \dots, \underline{b}_M, \underbrace{\underline{\xi}_1, \dots, \underline{\xi}_{m-M}}_{\text{generalized eigenvectors}}\}$ and dimension $\underline{\underline{m}}$.
  ↳ generalized eigenspace associated to λ.

- NOTE THAT $E_\lambda \subseteq \tilde{E}_\lambda$ SINCE IF

$$(A-\lambda I)\,\underline{v} = \underline{0} \quad \text{THEN} \quad (A-\lambda I)^m\,\underline{v} = 0 .$$

- IT CAN BE SHOWN THAT IF $(A-\lambda I)^p\,\underline{v} = \underline{0}$ FOR <u>ANY</u> $p \in \mathbb{N}$, THEN $\underline{v} \in \tilde{E}_\lambda$.

- $\underline{v}$ IS CALLED A <u>GENERALIZED EIGENVECTOR</u> (POWER VECTOR) IF $(A-\lambda I)^p\,\underline{v} = 0$ FOR SOME p. THE MINIMUM VALUE OF p SUCH THAT THIS IS TRUE IS CALLED THE <u>ORDER</u> OF $\underline{v}$. NOTE THAT IF $\underline{v}$ IS OF ORDER $p \geq 1$, THEN $\underline{w} = (A-\lambda I)\,\underline{v}$ IS OF ORDER $p-1$.

HOW TO FIND $\tilde{\mathcal{B}}_\lambda$? <u>DEFINE</u> $\underline{\xi}_i$ BY $(A-\lambda I)\,\underline{\xi}_i = \underline{\xi}_{i-1}$, $i=1,\ldots$ WHERE $\underline{\xi}_0 \doteq \underline{b}_M$.

$(A-\lambda I)\underline{\xi}_1 = \underline{b}_M$ ORDER 3

$$\tilde{\mathcal{B}}_\lambda = \{\underbrace{\underline{b}_1, \ldots, \underline{b}_M}_{\substack{\text{ORDER 1}\\ \text{(EIGENVECTORS)}}}, \underline{\xi}_1, \underline{\xi}_2, \ldots, \underline{\xi}_{m-M}\}$$

$(A-\lambda I)\underline{\xi}_2 = \underline{\xi}_1$

ORDER 2

ORDER $m-M+1$.

THEN, SINCE $(A-\lambda I)\,\underline{\xi}_i = \underline{\xi}_{i-1} \Rightarrow A\underline{\xi}_i = \lambda\underline{\xi}_i + \underline{\xi}_{i-1}$,

$$[A\underline{b}_i]_{\tilde{\mathcal{B}}_\lambda} = \lambda_i \underline{e}_i \quad \text{AND} \quad [A\underline{\xi}_i]_{\tilde{\mathcal{B}}_\lambda} = \lambda\,\underline{e}_{M+i} + \underline{e}_{M+i-1} .$$

$(i=1,\ldots,M)$ $\qquad$ $(i=1,\ldots,m-M)$

THIS IMPLIES THAT FOR ANY $A \in M_{n,n}$,

$$A = P\tilde{D}P^{-1} \quad \text{WHERE} \quad \tilde{D} = [A]_{\tilde{\beta}_\lambda} \quad \text{TAKES THE FORM}$$

$$\tilde{D} = \begin{pmatrix} \overbrace{\lambda_1 \ \cdots}^{M_1} & & & & & & & \\ & \overbrace{\boxed{\begin{matrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_1 & \ddots \ 1 \\ & & & \lambda_1 \end{matrix}}}^{m_1 - M_1} & & & & \\ & & \overbrace{\lambda_2 \ \cdots}^{M_2} & & \\ & & & \overbrace{\boxed{\begin{matrix} \lambda_2 & 1 & & \\ & \lambda_2 & 1 & \\ & & \lambda_2 & \ddots \ 1 \\ & & & \lambda_2 \end{matrix}}}^{m_2 - M_2} & \\ & & & & \ddots \end{pmatrix},$$

"JORDAN BLOCKS"

$$P = \Bigg(\overbrace{\underbrace{\begin{matrix}\text{EIGENVECTORS OF} \\ \lambda_1\end{matrix}}_{M_1} \ \vdots \ \underbrace{\begin{matrix}\text{GENERALIZED EIGENVECTORS OF} \\ \lambda_1\end{matrix}}_{m_1 - M_1}}^{\tilde{E}_{\lambda_1}} \ \Bigg| \ \cdots \ \Bigg| \ \overbrace{\underbrace{\begin{matrix}\text{EIGENVALUES OF} \\ \lambda_p\end{matrix}}_{M_p} \ \vdots \ \underbrace{\begin{matrix}\text{GENERALIZED} \\ \text{EIGENVECTORS OF} \\ \lambda_p\end{matrix}}_{m_p - M_p}}^{\tilde{E}_{\lambda_r}} \Bigg)$$

Ex. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (NOTE: WE SAW IN A PREVIOUS LECTURE THAT THIS MATRIX IS NOT DIAGONALIZABLE.)

$p_A(\lambda) = \lambda^2 = 0 \Rightarrow \lambda = 0$.

$\underline{\underline{E_0}}$: $A - 0I = \begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{\xi} = \xi_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \{(1,0)^T\}$ BASIS OF E_0.

I.E., $\lambda = 0$ HAS $m = 2$, BUT $M = 1$.

WHAT TO DO?

LOOK FOR GENERALIZED EIGENVECTORS.

$\underline{\underline{\tilde{E}_0}}$: HAS BASIS $\tilde{\mathcal{B}}_0 = \{\underline{b}, \underline{\xi}\}$, WHERE

$$\underline{b} = (1,0)^T \in E_0, \quad \text{AND}$$

$$(A - 0I)\underline{\xi} = \underline{b} \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \underline{\xi} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \underline{\xi} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

GENERALIZED EIGENVALUE OF ORDER 2.

SO, $\tilde{D} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$, $P = \begin{pmatrix} \underline{b} & \underline{\xi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$.

IN RETROSPECT, THIS IS OBVIOUS SINCE A IS ALREADY IN JORDAN FORM!

$\underline{\text{EX.}}$ $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$

$$\Rightarrow p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & -2 \\ -1 & -\lambda & 5 \\ -1 & -1 & 4-\lambda \end{vmatrix}$$

$$= (3-\lambda)\begin{vmatrix} -\lambda & 5 \\ -1 & 4-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ -1 & 4-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & -2 \\ -\lambda & 5 \end{vmatrix}$$

$$= (3-\lambda)\left[\lambda^2 - 4\lambda + 5\right] + (2-\lambda) + (2\lambda - 5)$$

$$= (\lambda - 3)\left[(-\lambda^2 + 4\lambda - 5) + 1\right]$$

$$= -(\lambda-3)(\lambda-2)^2 = 0 \Rightarrow \lambda = 3, \lambda = 2.$$

$\underline{\underline{E_3}}$: $A - 3I = \begin{pmatrix} 0 & 1 & -2 \\ -1 & -3 & 5 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \{(-1, 2, 1)^T\} \text{ BASIS OF } E_3.$$

$\underline{\underline{E_2}}$: $A - 2I = \begin{pmatrix} 1 & 1 & -2 \\ -1 & -2 & 5 \\ -1 & -1 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \underline{\xi} = \xi_3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \Rightarrow \{(-1, 3, 1)^T\} \text{ BASIS OF } E_2.$$

SO, $\lambda = 3$ HAS $\begin{cases} \text{ALG. MULT. } 1 \\ \text{GEOM. MULT. } 1 \end{cases}$

$\lambda = 2$ HAS $\begin{cases} \text{ALG. MULT. } 2 \\ \text{GEOM MULT. } 1 \end{cases} \Rightarrow A$ $\underline{\text{NOT}}$ DIAGONALIZABLE.

WHAT IS $\tilde{E}_2$?

$\underline{\underline{\tilde{E}_2}}$: HAS BASIS $\tilde{\mathcal{B}}_2 = (\underline{b}, \underline{\xi})$, WHERE $\underline{b} = (-1, 3, 1)^T \in E_2$.

$$(A - 2I)\underline{\xi} = \underline{b}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ -1 & -2 & 5 & 3 \\ -1 & -1 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 1 \\ 0 & \boxed{1} & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ $\xi_1 = 1 - \xi_3$
$\xi_2 = -2 + 3\xi_3$
ξ_3 FREE

$$\Rightarrow \underline{\xi} = \begin{pmatrix} +1 \\ -2 \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

LET $\underline{\xi_3} = 0$ (ONLY NEED $\underline{\text{ONE}}$ SOLN.)

$$\Rightarrow \underline{\xi} = (1, -2, 0)^T. \text{ (GENERALIZED EIGENVECTOR)}$$

THEREFORE,

$$A = P\tilde{D}P^{-1} \quad \text{WITH} \quad D = \begin{pmatrix} 3 & & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 3 & -2 \\ 1 & 1 & 0 \end{pmatrix}$$

LECTURE 15
02/22/12

TRICKS FOR FINDING / VERIFYING EIGENVALUES (4.6):

- IF $A \in M_{n,n}$ TRIANGULAR (UPPER- OR LOWER-TRIANGULAR), EIGENVALUES OF A ARE DIAGONAL ENTRIES $\{A_{ii}\}_{i=1}^{n}$.
- IF $A \in M_{n,n}(\mathbb{R})$ (I.E., A REAL MATRIX), $\lambda = a+ib$ EIGENVALUE $\Longleftrightarrow$ $\bar{\lambda} = a-ib$ EIGENVALUE.
- DEFINE TRACE $\mathrm{Tr}(A) \doteq \sum_{i=1}^{n} A_{ii}$. (SUM OF DIAGONAL ENTRIES). THEN, $\mathrm{Tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ (SUM OF EIGENVALUES, COUNTING MULTIPLICITIES).

  PF. EVEN THOUGH $AB \neq BA$ GENERALLY (I.E., MATRICES TYPICALLY DON'T COMMUTE), WE STILL HAVE THAT $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. THEN, SINCE A HAS JORDAN FORM $A = P\tilde{D}P^{-1}$,

$$\mathrm{Tr}(A) = \mathrm{Tr}(\underbrace{P\tilde{D}}_{\substack{\text{CALL}\\ B_1}}\underbrace{P^{-1}}_{\substack{\text{CALL}\\ B_2}}) = \mathrm{Tr}(\underbrace{P^{-1}}_{B_2}\underbrace{P\tilde{D}}_{B_1}) = \mathrm{Tr}(\tilde{D})$$
$$= \lambda_1 + \cdots + \lambda_n.$$

- $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$ (DETERMINANT IS PRODUCT OF EIGENVALUES)

  PF. SINCE $\det(AB) = \det(A)\det(B)$ FOR ANY $A, B \in M_{n,n}$,

$$\det(A) = \det(P\tilde{D}P^{-1}) = \cancel{\det(P)}\det(\tilde{D})\underbrace{\cancel{\det(P^{-1})}}_{=\frac{1}{\det(P)}} = \det(\tilde{D})$$
$$= \lambda_1 \cdots \lambda_n.$$

Ex. What are eigenvalues of $A = \begin{pmatrix} -6 & 7 \\ 4 & 6 \end{pmatrix}$?

$$\left.\begin{array}{l} Tr(A) = 0 = \lambda_1 + \lambda_2 \\ \det(A) = -36 - 28 = -64 = \lambda_1 \lambda_2 \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_1 = 8 \\ \lambda_2 = -8 \end{array}.$$

Note: For most problems, use these tricks to <u>verify</u> your answers are correct.

Evolution problems (5.1, 5.2):

① **Discrete-time evolution:**

$$\left\{ \begin{array}{l} \underline{x}(k) = A\,\underline{x}(k-1) \\ \underline{x}(0) \text{ known.} \end{array} \right. \quad , \quad \underbrace{\underline{x}}_{\text{state of system at time } k \in \mathbb{N}}(k) \in \mathbb{R}^n \text{ for all } k = 0, 1, 2, \ldots$$

given.

$$\underline{x}(k) = \underbrace{(x_1(k), x_2(k), \ldots, x_n(k))^T}_{\text{components of state vector } \underline{x}(k)}$$

$\boxed{\text{Sol'n:}}$ $\underline{x}(k) = A\,\underline{x}(k-1) = A(A\,x(k-2)) = \ldots = \underbrace{A^k}_{\text{needs to be determined}}\,\underline{x}(0).$

② **Continuous-time evolution:**

$$\left\{ \begin{array}{l} \dfrac{d\underline{x}(t)}{dt} = A\,\underline{x}(t) \\ \underline{x}(0) \text{ known.} \end{array} \right. \quad , \quad \underbrace{\underline{x}}_{\text{state of system at time } t \in [0,\infty)}(t) \in \mathbb{R}^n \text{ for all } t \geq 0.$$

given.

$$\underline{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T.$$

This is known as ~~a~~ <u>system</u> of first-order ordinary differential equations (ODE).

$\boxed{\text{SOL'N:}}$ IF $n=1$, I.E, IF A IS A SCALAR, SAY a, THEN THE EVOLUTION PROBLEM IS EASILY SOLVED.

$$\begin{cases} \frac{dx(t)}{dt} = a\,x(t) \\ x(0) \text{ known} \end{cases} \Rightarrow \int \frac{dx(t)}{x(t)} = \int a\,dt \Rightarrow \ln(x(t)) - \ln(x(0)) = ta$$

$$\Rightarrow x(t) = e^{ta}x(0).$$

(TO CHECK THIS IS CORRECT, SUBSTITUTE INTO THE ODE TO FIND THAT $\frac{d}{dt}(\underbrace{e^{ta}x(0)}_{x(t)}) = a e^{ta}x(0) = a x(t)$. ✓)

BY ANALOGY, IF $n>1$ THEN $A \in M_{n,n}$ IS A MATRIX AND THE SOLN. IS

$$\underline{x}(t) = \underbrace{e^{tA}}\,\underline{x}(0).$$

WE WILL DEFINE THIS LATER USING MATRIX EXPONENTIALS, WHICH WILL DETERMINE THIS TO BE

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{A^k}.$$

NEEDS TO BE DETERMINED.

NOTE: FOR BOTH DISCRETE- AND CONTINUOUS-TIME EVOLUTIONS WE NEED TO FIND A^k, FOR SOME GIVEN $k \in \mathbb{N}$. FOR LARGE k, THIS CAN BE DIFFICULT — UNLESS WE DIAGONALIZE A! IF A IS DIAGONALIZABLE, THEN

$$A^k = (PDP^{-1})^k = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})$$

$$= PD^kP^{-1}, \text{ WHERE } D^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}.$$

LECTURE 16
02/24/12

DISCRETE-TIME EVOLUTION:

EX (POPULATION GROWTH, FIBONACCI SEQUENCE)

LET $x_1(t)$, $x_2(t)$ BE THE NUMBER OF JUVENILE AND ADULT RABBITS THAT LIVE IN A CERTAIN REGION. THESE POPULATIONS EVOLVE ACCORDING TO THE FOLLOWING RULES:

- EVERY MONTH, EACH ADULT GIVES BIRTH TO ONE JUVENILE.
- JUVENILES GROW INTO ADULTS IN ONE MONTH
- ADULTS / JUVENILES DO NOT DIE.

THEN, FOR EACH MONTH $k = 0, 1, 2, \ldots$,

$$\Delta x_1(k) \doteq x_1(k) - x_1(k-1) = \overbrace{x_2(k-1)}^{\text{BIRTH OF NEW JUVENILES}} - \overbrace{x_1(k-1)}^{\text{JUVENILES BECOME ADULTS}}$$

$$\Delta x_2(k) \doteq x_2(k) - x_2(k-1) = \overbrace{x_1(k-1)}^{\text{JUVENILES BECOME ADULTS}}.$$

LETTING $\underline{x}(k) = (x_1(k), x_2(k))^T$, WE HAVE THAT.

$$\underline{x}(k) = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \underline{x}(k-1).$$

$$\Rightarrow p_A(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda_1 = \underbrace{\frac{1+\sqrt{5}}{2}}_{\approx 1.618 \text{ ``GOLDEN RATIO''}}, \quad \lambda_2 = \underbrace{\frac{1-\sqrt{5}}{2}}_{\approx -0.618}.$$

THE CORRESPONDING EIGENSPACES ARE

$\underline{\underline{E_{\frac{1+\sqrt{5}}{2}}}}$:

$$\begin{pmatrix} -\left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \times \frac{1-\sqrt{5}}{2} \rightarrow \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} -R_1$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \left(-(1-\sqrt{5}), 2\right)^T \right\} \text{ BASIS.}$$

$\underline{\underline{E_{\frac{1-\sqrt{5}}{2}}}}$:

$$\begin{pmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & 1 \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \times \frac{1+\sqrt{5}}{2} \rightarrow \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} -R_1$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \left(-(1+\sqrt{5}), 2\right)^T \right\} \text{ BASIS.}$$

LET $\begin{cases} \underline{b}_1 = \left(-(1-\sqrt{5}), 2\right)^T \\ \underline{b}_2 = \left(-(1+\sqrt{5}), 2\right)^T \end{cases}$ BE THE BASIS $\mathcal{B}$ OF EIGENVECTORS OF A.

THEN,

$$\begin{array}{ccc} (a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}} & \xrightarrow{D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}} & [\underline{x}(k)]_{\mathcal{B}} = (a_1(k), a_2(k))^T \\ \Big\uparrow P^{-1} & & \Big\downarrow P = (\underline{b}_1\ \underline{b}_2) \\ (x_1(0), x_2(0))^T = \underline{x}(0) & \dashrightarrow_{A^k} & \underline{x}(k) = (x_1(k), x_2(k))^T. \end{array}$$

THAT IS, $\underline{x}(k) = \underbrace{a_1(k)}_{= \lambda_1^k a_1(0)} \underline{b}_1 + \underbrace{a_2(k)}_{= \lambda_2^k a_2(0)} \underline{b}_2$, WITH $\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = [\underline{x}(0)]_{\mathcal{B}}$

FOR EXAMPLE, SUPPOSE WE START WITH ONE JUVENILE AND NO ADULTS. THEN,

$$\underline{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow [\underline{x}(0)]_{\mathcal{B}} = \Big(\underbrace{\tfrac{1}{2\sqrt{5}}}_{a_1(0)} , \underbrace{-\tfrac{1}{2\sqrt{5}}}_{a_2(0)} \Big)^T .$$

$$\Rightarrow \underline{x}(k) = \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^k}_{\to \infty \text{ AS } k\to\infty} \frac{1}{2\sqrt{5}} \underline{b}_1 + \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^k}_{\to 0 \text{ AS } k \to \infty} \left(-\frac{1}{2\sqrt{5}}\right) \underline{b}_2$$

$\Rightarrow \underline{b}_1$ "UNSTABLE MODE" $\qquad \Rightarrow \underline{b}_2$ "STABLE MODE".

FOR k LARGE

$$\approx \left(\frac{1+\sqrt{5}}{2}\right)^k \frac{1}{2\sqrt{5}} \underline{b}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{k} \end{pmatrix} .$$

CONSEQUENCES (FOR LARGE k):

- POP. OF ADULTS WILL BE APPROX. $\frac{1+\sqrt{5}}{2} \approx 1.618$ TIMES MORE THAN THAT OF JUVENILES.
- BOTH POPULATIONS GROW EXPONENTIALLY, APPROX. BY 1.618 TIMES EACH MONTH.
- TOTAL POPULATION OF RABBITS IS APPROX.

$$p(k) = x_1(k) + x_2(k) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} .$$

CONTINUOUS-TIME EVOLUTION:

LAST TIME, WE SAW THAT THE SOLN. TO

$$\begin{cases} \dfrac{d\underline{x}}{dt} = A\underline{x} \\ \underline{x}(0) \text{ given} \end{cases} \quad \text{is} \quad \underline{x}(t) = e^{tA}\,\underline{x}(0).$$

WHAT IS THIS?

MATRIX EXPONENTIAL (4.8):

FOR SCALAR $a \in \mathbb{R}$, THE POWER SERIES OF e^a IS

$$e^a = 1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{a^k}{k!}.$$

FOR $A \in M_{n,n}$, DEFINE THE MATRIX EXPONENTIAL

$$e^A \doteq I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

(FOR $L: V \to V$, CAN DEFINE $L^n \doteq L \circ L \circ \cdots \circ L$ (n TIMES)
AND $e^L = I + \frac{L}{1!} + \frac{L^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{L^k}{k!}$.)

- IF A IS DIAGONALIZABLE, $A = PDP^{-1}$ AND

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} (PD^kP^{-1})$$

$$= P\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k\right) P^{-1} = P e^{tD} P^{-1}.$$

NOTE THAT FOR D DIAGONAL,

$$e^{tD} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}.$$

THEREFORE, IF A IS DIAGONALIZABLE,

$$\underline{x}(t) = e^{tA}\,\underline{x}(0)$$

$$= P\,e^{tD}\,P^{-1}\,\underline{x}(0)$$

$$\begin{tikzcd}[column sep=huge, row sep=large]
{[\underline{x}(0)]_{\mathcal{B}}} \arrow[r, "{e^{tD} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}}"'] & {[\underline{x}(t)]_{\mathcal{B}}} \arrow[d, "{P = (\underline{b}_1 \cdots \underline{b}_n)}"] \\
\underline{x}(0) \arrow[u, "P^{-1}"] \arrow[r, dashed, "e^{tA}"'] & \underline{x}(t)
\end{tikzcd}$$

WHERE $\{\underline{b}_1, \dots, \underline{b}_n\} = \mathcal{B}$ IS THE BASIS OF EIGENVECTORS.