Lecture 17
02/27/12

Recall:

① **Discrete-time evolution**

$$\begin{cases} \underline{x}(k) = A\,\underline{x}(k-1) \\ \underline{x}(0) \text{ known} \end{cases} \implies \begin{aligned} \underline{x}(k) &= A^k\,\underline{x}(0) \\ &= P D^k P^{-1}\,\underline{x}(0) \end{aligned}$$

if $A = PDP^{-1}$.

Ex. $(n=2)$

$$(a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}}$$

$$\begin{tikzcd} {[\underline{x}(0)]_{\mathcal{B}}} \arrow[r, "D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}"] & {[\underline{x}(k)]_{\mathcal{B}}} \arrow[d, "P"] \\ \underline{x}(0) \arrow[u, "P^{-1}"] \arrow[r, "A^k"] & \underline{x}(k) \end{tikzcd}$$

$$\underline{x}(0) = (x_1(0), x_2(0))^T$$

$$[\underline{x}(k)]_{\mathcal{B}} = (a_1(k), a_2(k))^T = (\lambda_1^k a_1(0), \lambda_2^k a_2(0))^T.$$

with $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ basis of e-vectors of A.

② **Cont. time evolution**

$$\begin{cases} \frac{dx(t)}{dt} = A\,x(t) \\ x(0) \text{ known} \end{cases} \implies \begin{aligned} \underline{x}(t) &= e^{tA}\,\underline{x}(0) \\ &= P e^{tD} P^{-1}\,\underline{x}(0) \end{aligned}$$

if $A = PDP^{-1}$.

Ex. $(n=2)$

$$(a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}}$$

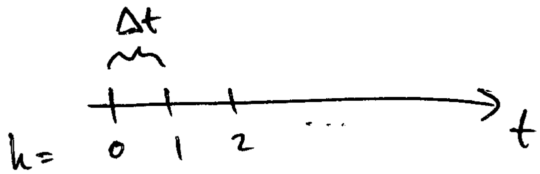
$$\begin{tikzcd} {[\underline{x}(0)]_{\mathcal{B}}} \arrow[r, "e^{tD} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}"] & {[\underline{x}(t)]_{\mathcal{B}}} \arrow[d, "P"] \\ \underline{x}(0) \arrow[u, "P^{-1}"] \arrow[r, "e^{tA}"] & \underline{x}(t) \end{tikzcd}$$

$$\underline{x}(0) = (x_1(0), x_2(0))^T$$

$$[\underline{x}(t)]_{\mathcal{B}} = (a_1(t), a_2(t))^T = (e^{t\lambda_1} a_1(0), e^{t\lambda_2} a_2(0))^T$$

with $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ basis of e-vectors of A

NOTE: (2) IS A LIMIT OF (1).



RESTRICT TIMES TO $t = k\Delta t,\ k \in \mathbb{N}$

$$\Rightarrow \quad \frac{d\underline{x}(t)}{dt} \approx \frac{\underline{x}((k+1)\Delta t) - \underline{x}(k\Delta t)}{\Delta t}$$

$$\begin{array}{c} \shortparallel \\ A\underline{x}(t) \approx A\,\underline{x}(k\Delta t). \end{array}$$

$$\Rightarrow \begin{cases} \underline{x}((k+1)\Delta t) = (I + (\Delta t)A)\,\underline{x}(k\Delta t) \\ \underline{x}(0) \ \text{known} \end{cases}$$

$$\Rightarrow \quad \underline{x}(k\Delta t) = (I + (\Delta t)A)^k\,\underline{x}(0).$$

NOTE NOW THAT AS $\Delta t \to 0$,

$$(I + (\Delta t)A)^k = (I + (\Delta t)A)^{\frac{k\Delta t}{\Delta t}}$$

$$= (I + (\Delta t)A)^{\frac{t}{\Delta t}}$$

$$\xrightarrow[\Delta t \to 0]{} e^{tA}$$

(THIS IS THE ANALOGUE OF THE DEFINITION OF THE EXPONENTIAL:

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x.$$)

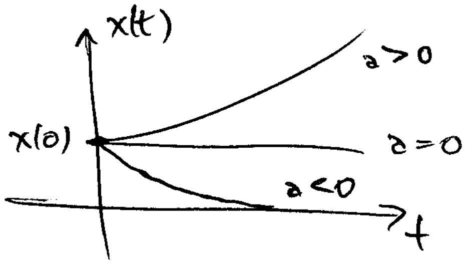
THAT IS, WE CAN THINK OF CONTINUOUS-TIME SYSTEMS (SYSTEMS OF LINEAR ODE) AS LIMITS OF DISCRETE-TIME SYSTEMS.

EX. (SCALAR EXPONENTIAL GROWTH / DECAY WITH RATE a)

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) \text{ GIVEN} \end{cases} \Rightarrow \quad \mathbf{x}(t) = e^{ta}\, x(0)$$

$$\xrightarrow{t \to \infty} \begin{cases} \pm\infty, & a > 0 \\ 0, & a < 0 \\ x(0), & a = 0 \end{cases}$$

IF $x(0) \neq 0$.



EX. (RADIOACTIVE DECAY W/ MULTIPLE POPULATIONS)

$$X \xrightarrow{r_1} Y, \quad Y \xrightarrow{r_2} Z \qquad \text{WITH } r_1, r_2 \geq 0, \; r_1 \neq r_2.$$

$$\Rightarrow \begin{cases} \dot{x}_1 = -r_1 x_1 \\ \dot{x}_2 = r_1 x_1 - r_2 x_2 \end{cases}$$

WHERE $x_1(t), x_2(t)$ ARE POPULATIONS OF X, Y AT TIME $t \geq 0$, AND $\bullet = \dfrac{d}{dt}$.

$$\Rightarrow \frac{d\underline{x}(t)}{dt} = A\,\underline{x}(t) \qquad \text{WITH} \quad A = \begin{pmatrix} -r_1 & 0 \\ r_1 & -r_2 \end{pmatrix}$$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

E-VALUES / E-VECTORS ARE

$$\lambda_1 = -r_1, \quad E_{-r_1} = \{\overbrace{((r_2 - r_1), r_1)^T}^{\underline{b}_1}\}$$

$$\lambda_2 = -r_2, \quad E_{-r_2} = \{\underbrace{(0, 1)^T}_{\underline{b}_2}\}.$$

SUPPOSE $\underline{x}(0) = (1,0)^T$. THEN,

$$\underline{x}(0) = \frac{1}{r_2 - r_1}\underline{b}_1 - \frac{r_1}{r_2 - r_1}\underline{b}_2$$

AND $$\underline{x}(t) = \frac{e^{-r_1 t}}{r_2 - r_1}\underline{b}_1 - \frac{r_1 e^{-r_2 t}}{r_2 - r_1}\underline{b}_2$$

$$= \Big(\underbrace{e^{-r_1 t}}_{x_1(t)}, \underbrace{\frac{r_1}{r_2 - r_1}\left(e^{-r_1 t} - e^{-r_2 t}\right)}_{x_2(t)}\Big)^T.$$

NOTE THAT AS $r_1 \to r_2$ WE CAN STILL RECOVER A SOLUTION FOR $\underline{x}(t)$:

$$\lim_{r_1, r_2 \to r} \underline{x}(t) = \left(e^{-rt}, rt\, e^{-rt}\right)^T.$$

IN THIS CASE, A IS NOT DIAGONALIZABLE WHEN $r_1 = r_2 = r$, BUT IT IS ALMOST DIAGONALIZABLE.

LECTURE 18
02/29/12

Q: FOR EVOLUTION PROBLEMS, WHAT IF WE HAVE COMPLEX EIGENVALUES?

A: SAME METHODS, BUT WILL SEE OSCILLATIONS IN SYSTEM ALONG W/ EXPONENTIAL DECAY / GROWTH!

EX
$$\begin{cases} \dfrac{d\underline{x}(t)}{dt} = A\underline{x}(t) \\ \underline{x}(0) \text{ GIVEN} \end{cases}, \quad A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

E-VALUES / E-VECTORS OF A:
$$\lambda = 1 \pm 2i$$
$$\underline{b} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\underline{v}} \pm i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\underline{w}}$$

$$\Rightarrow \underline{x}(t) = e^{tA}\underline{x}(0) = \underbrace{Pe^{tD}P^{-1}}_{\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -1 \end{pmatrix}^{-1}} \underline{x}(0)$$

WE USE EULER'S FORMULA $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, $\theta \in \mathbb{R}$, TO GET THAT

$$\begin{pmatrix} e^{(1+2i)t} & 0 \\ 0 & e^{(1-2i)t} \end{pmatrix} = \begin{pmatrix} e^{t}(\cos(2t) + i\sin(2t)) & 0 \\ 0 & e^{t}(\cos(2t) - i\sin(2t)) \end{pmatrix}$$

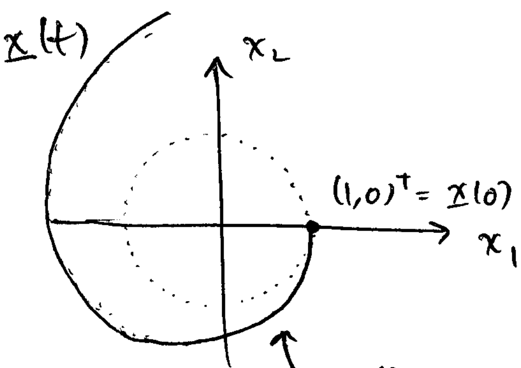
SINCE $\cos(-x) = \cos(x)$, $\sin(-x) = -\sin(x)$.

THEN, THE SOLUTION IS

$$\underline{x}(t) = e^{t} \begin{pmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{pmatrix} \underline{x}(0)$$

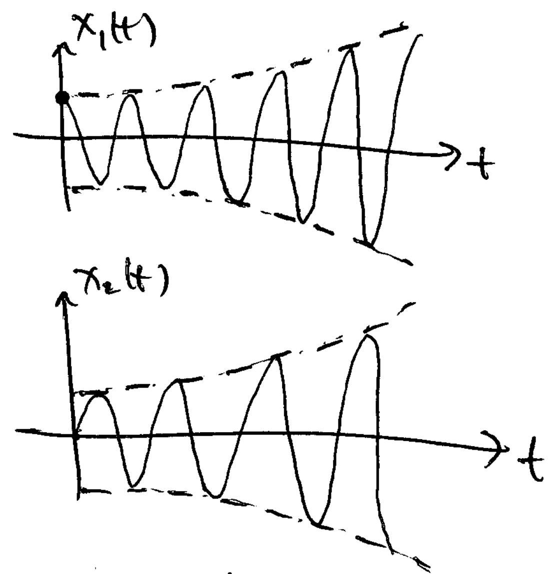
<u>REMARK</u>: REMEMBER, SINCE A IS REAL WE MUST HAVE THAT e^{tA} IS REAL AND $Pe^{tD}P^{-1}$ IS REAL AS WELL! SO THE FINAL ANSWER SHOULD CONSIST ONLY OF REAL TERMS.

WE CAN PLOT THE SOLUTION IN SEVERAL WAYS (GIVEN AN INITIAL CONDITION $\underline{x}(0)$ — FOR EX., $\underline{x}(0) = (1,0)^T$):



SPIRAL OUTWARD SINCE HAVE TERM e^{t}.

<u>PHASE PLANE</u>.



$$x_1(t) = e^{t}\cos(2t)$$
$$x_2(t) = -e^{t}\sin(2t).$$

- WE NOTE THAT THE REAL PART OF THE PAIR OF COMPLEX EIGENVALUES $\lambda = 1 \pm 2i$ DETERMINES THE <u>RATE</u> OF <u>GROWTH / DECAY</u>, WHILE THE IMAGINARY PART DETERMINES THE <u>FREQUENCY</u> OF <u>OSCILLATION</u>.

FOR EXAMPLE, IF $\lambda = -1 \pm 2i$ INSTEAD, WE WOULD HAVE HAD A SOLUTION w/ EXPONENTIAL <u>DECAY</u> e^{-t} INSTEAD OF GROWTH (A SPIRAL INWARD IN THE PHASE PLANE) BUT THE SAME FREQUENCY OF OSCILLATION.

<u>REMARK</u>: EVERY n^{th}-ORDER, HOMOGENEOUS, CONST. COEFF. SCALAR ODE

$$(\star) \quad \frac{d^n y}{dt^n} + c_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + c_0 y = 0, \quad \begin{array}{l} y(t) \in \mathbb{R} \\ t \geq 0. \\ c_i \in \mathbb{R} \text{ const.} \end{array}$$

CAN BE WRITTEN AS A <u>SYSTEM</u> OF 1^{st}-ORDER ODE. TO SEE THIS, LET

$$\underline{x}(t) = \begin{pmatrix} x_0(t) \\ x_1(t) \\ \vdots \\ x_{n-1}(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ \frac{dy}{dt}(t) \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(t) \end{pmatrix} \in \mathbb{R}^n \text{ FOR ALL } t \geq 0.$$

THEN,

$$\frac{d\underline{x}(t)}{dt} = \begin{pmatrix} \frac{dy}{dt}(t) \\ \frac{d^2y}{dt^2}(t) \\ \vdots \\ \frac{d^n y}{dt^n}(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{n-1}(t) \\ -c_{n-1}x_{n-1}(t) - \cdots - c_0 x_0(t) \end{pmatrix}$$

WHERE WE HAVE USED $(\star)$ TO REWRITE $\frac{d^n y}{dt^n}$ IN TERMS OF $y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}y}{dt^{n-1}}$. THEREFORE,

$$\frac{d\underline{x}}{dt} = A\underline{x}, \quad A = \left(\begin{array}{c|ccc} 0 & & I_{n-1} & \\ \hline -c_0 & -c_1 & \cdots & -c_{n-1} \end{array}\right) \updownarrow n$$

$$\overset{\longleftrightarrow}{n}$$

w/ I_{n-1} AN $(n-1)\times(n-1)$ IDENTITY MATRIX.

<u>NOTE:</u> TO SOLVE THIS LINEAR EVOLUTION SYSTEM, WE NEED TO BE GIVEN

$$\underline{x}(0) = \left.\begin{pmatrix} y(0) \\ \frac{dy}{dt}(0) \\ \vdots \\ \frac{d^{n-1}y}{dt^{n-1}}(0) \end{pmatrix}\right\} \text{ I.E., } n \text{ INITIAL CONDITIONS.}$$

<u>EX.</u> $\begin{cases} y'' + by' + y = 0 \\ \text{w/ } y(0),\ y'(0) \text{ GIVEN.} \end{cases}$ "$' = \frac{d}{dt}$".

LET $\underline{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$. THEN, $\underline{x}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} y \\ y' \end{pmatrix}}_{\underline{x}}$.

$p_A(\lambda) = \lambda^2 + b\lambda + 1$

$\rightsquigarrow$ E-VALUES OF A: $\lambda = \dfrac{-b \pm \sqrt{b^2-4}}{2}$.

SO, IF $|b| < 2$ WE WILL GET OSCILLATIONS ALONG w/ EXPONENTIAL GROWTH/DECAY IN SOLUTION $y(t)$.

LECTURE 19
03/02/12

Q: WHAT IS THE LONG-TIME BEHAVIOR OF A LINEAR EVOLUTION SYSTEM? HOW CAN WE DETERMINE IT WITHOUT HAVING TO SOLVE IT EXPLICITLY?

STABILITY AND LONG-TIME BEHAVIOR (5.5):

ASSUME $A = PDP^{-1}$, $D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$. THEN,

DISCRETE-TIME EVOLUTION

$$\underline{x}(k) = A^k \underline{x}(0) = PD^kP^{-1}\underline{x}(0) = \lambda_1^k a_1(0)\underline{b}_1 + \dots + \lambda_n^k a_n(0)\underline{b}_n$$

WITH $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ BASIS OF E-VECTORS.

$\Rightarrow$ LONG-TIME BEHAVIOR DICTATED BY $\lambda_1^k, \dots, \lambda_n^k$.

CONT.-TIME EVOLUTION

$$\underline{x}(t) = e^{tA}\underline{x}(0) = Pe^{tD}P^{-1}\underline{x}(0) = e^{\lambda_1 t}a_1(0)\underline{b}_1 + \dots + e^{\lambda_n t}a_n(0)\underline{b}_n.$$

$\Rightarrow$ LONG-TIME BEHAVIOR DICTATED BY $e^{\lambda_1 t}, \dots, e^{\lambda_n t}$.

NOTE: IF $z \in \mathbb{C}$, $z = a + ib$ w/ $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$
$= re^{i\theta}$ w/ $r = \sqrt{a^2+b^2}$, $\theta = \arctan\left(\frac{b}{a}\right)$.

THE MAGNITUDE OF z IS $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2} = r$.

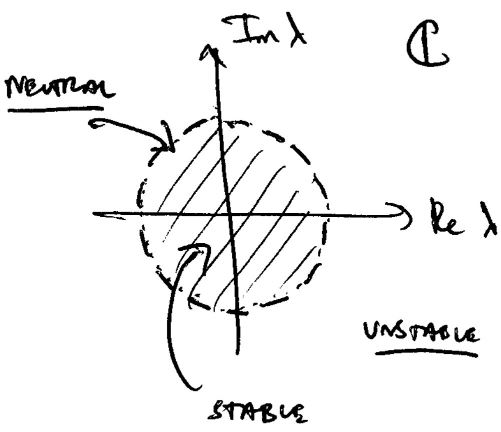
THEN,

$$\begin{cases} |z|>1 \Rightarrow z^k \to \infty \\ |z|=1 \Rightarrow |z^k| = 1 \quad \text{AS } k\to\infty \\ |z|<1 \Rightarrow z^k \to 0 \end{cases}$$

$$\begin{cases} a = Re(z) > 0 \Rightarrow e^{zt} \to \infty \\ a = Re(z) = 0 \Rightarrow |e^{zt}| = |e^{ibt}| = 1 \quad \text{AS } t\to\infty. \\ a = Re(z) < 0 \Rightarrow e^{zt} \to 0 \end{cases}$$

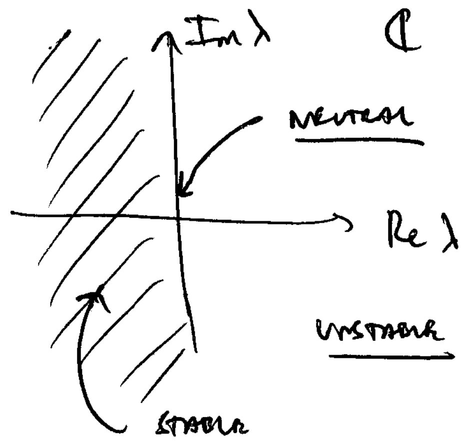
THEREFORE,

<u>DISCRETE - TIME</u>



- $|\lambda_i| > 1 \Rightarrow \underline{b}_i$ <u>UNSTABLE</u> MODE
  $|\lambda_i| = 1 \Rightarrow \underline{b}_i$ <u>NEUTRAL</u> MODE
  $|\lambda_i| < 1 \Rightarrow \underline{b}_i$ <u>STABLE</u> MODE

<u>CONT. - TIME.</u>



- $Re(\lambda_i) > 0 \Rightarrow \underline{b}_i$ <u>UNSTABLE</u> MODE
  $Re(\lambda_i) = 0 \Rightarrow \underline{b}_i$ <u>NEUTRAL</u> MODE
  $Re(\lambda_i) < 0 \Rightarrow \underline{b}_i$ <u>STABLE</u> MODE.

- ASSUME $\lambda_1, \dots, \lambda_r$ ARE <u>DISTINCT</u> E-VALUES OF A,

SO $$\underline{x}(k) = \lambda_1^k \underline{d}_1 + \dots + \lambda_r^k \underline{d}_r \quad \text{(DISCRETE-TIME)}$$
$$\underline{x}(t) = e^{\lambda_1 t} \underline{d}_1 + \dots + e^{\lambda_r t} \underline{d}_r \quad \text{(CONT.-TIME)}.$$

FOR SOME VECTORS $\underline{d}_1, \dots, \underline{d}_r \in \mathbb{R}^n$.

ARRANGE E-VALUES IN ORDER OF DECREASING MAGNITUDE:

$$\rho(A) \doteq |\lambda_1| > |\lambda_2| > \dots > |\lambda_r|.$$

<u>SPECTRAL RADIUS</u> OF A

λ_1 IS THE <u>DOMINANT</u> E-VALUE OF SYSTEM.
λ_2 DETERMINES CONVERGENCE RATE TO "EQUILIBRIUM",

SINCE $$\left\| \frac{\underline{x}(k)}{\lambda_1^k} - \underline{d}_1 \right\| = \left\| \left(\frac{\lambda_2}{\lambda_1}\right)^k \underline{d}_2 + \dots \right\| \lessapprox c \left(\frac{\lambda_2}{\lambda_1}\right)^k \quad \text{AS } k \to \infty$$

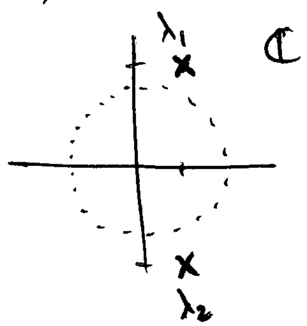
(SIMILAR CALCULATION IN CONT. TIME CASE.).

- STABILITY DETERMINED BY DOMINANT E-VALUE.

λ_1 STABLE $\Rightarrow$ SYSTEM <u>STABLE</u>
λ_1 NEUTRAL $\Rightarrow$ SYSTEM <u>NEUTRALLY STABLE</u>
λ_1 UNSTABLE $\Rightarrow$ SYSTEM <u>UNSTABLE</u>

<u>EX.</u> $$\begin{cases} \underline{x}(k) = A\,\underline{x}(k-1) \\ \underline{x}(0) \text{ GIVEN.} \end{cases}$$, A HAS EVALUES $\lambda = \frac{1}{2} \pm \frac{4}{3} i$.

THEN, SPECTRUM OF A IS:



SINCE $|\lambda_1| = |\lambda_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{4}{3}\right)^2} > 1$, THE SYSTEM IS <u>UNSTABLE</u>.

<u>EX.</u> $\begin{cases} \frac{dx}{dt} = A\underline{x} \\ \underline{x}(0) \text{ GIVEN} \end{cases}$, A HAS E-VALUES $\lambda = \frac{1}{2} \pm \frac{4}{3} i$.

SINCE $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) = \frac{1}{2} > 0$, THE SYSTEM IS <u>UNSTABLE</u>.