LECTURE 23
03/19/12

Q: UP TILL NOW, WE HAVE NOT DISCUSSED GEOMETRY OF VECTOR SPACES (LENGTH OF VECTOR, ANGLE BETWEEN VECTORS, ETC.). IF WE IMPOSE GEOMETRIC STRUCTURE, WHAT DOES THIS IMPLY ABOUT LINEAR OPERATORS?

REAL AND COMPLEX INNER PRODUCT SPACES (6.1–6.2)

EX. (STANDARD REAL EUCLIDEAN SPACE)

REAL VECTOR SPACE → $V = \mathbb{R}^n$ WITH STANDARD INNER PRODUCT (DOT PRODUCT) $\langle \underline{x} | \underline{y} \rangle = x_1 y_1 + \cdots + x_n y_n \; (= \underline{x} \cdot \underline{y}) = \underline{x}^T \underline{y}$

WHERE $\underline{x} = (x_1, \dots, x_n)^T$, $\underline{y} = (y_1, \dots, y_n)^T$, $x_i, y_i \in \mathbb{R}$.

$$\|\underline{x}\| = \sqrt{\langle \underline{x} | \underline{x} \rangle} = \sqrt{x_1^2 + \cdots + x_n^2}$$

LENGTH OF $\underline{x}$,

$$\cos\theta = \frac{\langle \underline{x} | \underline{y} \rangle}{\|\underline{x}\| \, \|\underline{y}\|}$$

ANGLE BETWEEN $\underline{x}, \underline{y}$.

EX (STANDARD COMPLEX EUCLIDEAN SPACE)

COMPLEX VECTOR SPACE → $V = \mathbb{C}^n$ WITH STANDARD INNER PRODUCT $\langle \underline{w} | \underline{z} \rangle = \overline{w_1} z_1 + \cdots + \overline{w_n} z_n$
$= \overline{\underline{w}}^T \underline{z}$
$= \underline{w}^* \underline{z}$

WHERE $\underline{w} = (w_1, \dots, w_n)^T$, $\underline{z} = (z_1, \dots, z_n)^T$, $w_i, z_i \in \mathbb{C}$.

(WE DENOTE $A^* = (\overline{A})^T$ FOR ANY $A \in M_{n,m}(\mathbb{C})$.)

$$\|\underline{w}\| = \sqrt{\langle \underline{w} | \underline{w} \rangle} = \sqrt{\overline{w_1} w_1 + \cdots + \overline{w_n} w_n}.$$

IN GENERAL:

DEF. FOR REAL VECTOR SPACE V, $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{R}$
IS AN INNER PRODUCT IF FOR ALL $\underline{x}, \underline{y} \in V$,

(i) SYMMETRY: $\langle \underline{x} | \underline{y} \rangle = \langle \underline{y} | \underline{x} \rangle$.

(ii) BILINEARITY: $\langle c_1 \underline{x} + c_2 \underline{y} | \underline{z} \rangle = c_1 \langle \underline{x} | \underline{z} \rangle + c_2 \langle \underline{y} | \underline{z} \rangle$
$\langle \underline{x} | c_1 \underline{y} + c_2 \underline{z} \rangle = c_1 \langle \underline{x} | \underline{y} \rangle + c_2 \langle \underline{x} | \underline{z} \rangle$.

(iii) POSITIVITY: $\langle \underline{x} | \underline{x} \rangle = 0 \iff \underline{x} = \underline{0}$.
AND $\langle \underline{x} | \underline{x} \rangle > 0$ OTHERWISE.

- LENGTH OF $\underline{x} \in V$: $\|\underline{x}\| = \sqrt{\langle \underline{x} | \underline{x} \rangle}$.
- ANGLE BETWEEN $\underline{x}, \underline{y} \in V$: $\cos \theta = \dfrac{\langle \underline{x} | \underline{y} \rangle}{\|\underline{x}\| \|\underline{y}\|}$.

DEF. FOR COMPLEX ~~VECTOR~~ SPACE V, $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{C}$
IS AN INNER PRODUCT IF FOR ALL $\underline{x}, \underline{y} \in V$,

"HERMITIAN"
(i) CONJUGATE SYMMETRY: $\langle \underline{x} | \underline{y} \rangle = \overline{\langle \underline{y} | \underline{x} \rangle}$.

(ii) SESQUILINEARITY: $\langle c_1 \underline{x} + c_2 \underline{y} | \underline{z} \rangle = \bar{c}_1 \langle \underline{x} | \underline{z} \rangle + \bar{c}_2 \langle \underline{y} | \underline{z} \rangle$
$\langle \underline{x} | c_1 \underline{y} + c_2 \underline{z} \rangle = c_1 \langle \underline{x} | \underline{y} \rangle + \bar{c}_2 \langle \underline{x} | \underline{z} \rangle$.

(iii) POSITIVITY: $\langle \underline{x} | \underline{x} \rangle > 0$ FOR ALL $\underline{x} \neq 0$.
(REAL NUMBER DUE TO CONJUGATE SYMMETRY)

- LENGTH OF $\underline{x} \in V$: $\|\underline{x}\| = \sqrt{\langle \underline{x} | \underline{x} \rangle}$.

NOTE: REAL INNER PRODUCT SPACES CAN BE SEEN AS A SPECIAL CASE OF COMPLEX INNER PRODUCT SPACES !

EX. $V = \mathbb{R}^n$, $\langle \underline{x} | \underline{y} \rangle \doteq x_1 y_1 + \cdots + x_n y_n$.

IF $n=3$, WHAT IS $\langle \underbrace{(3,1,2)^T}_{\underline{x}} | \underbrace{(-1,6,0)^T}_{\underline{y}} \rangle$?

$\langle \underline{x} | \underline{y} \rangle = -3 + 6 + 0 = 3$.

EX. $V = \mathbb{R}^n$, $\langle \underline{x} | \underline{y} \rangle \doteq x_1 y_1 + 2 x_2 y_2 + \cdots + n x_n y_n$.

CAN CHECK THAT THIS IS INDEED AN INNER PRODUCT.

IF $n=3$, WHAT IS $\langle \underbrace{(3,1,2)^T}_{\underline{x}} | \underbrace{(-1,6,0)^T}_{\underline{y}} \rangle$?

$\langle \underline{x} | \underline{y} \rangle = -3 + 2 \cdot 6 + 3 \cdot 0 = 9$.

EX. $V = \mathbb{R}_2[t]$, $\langle a_0 + a_1 t + a_2 t^2 | b_0 + b_1 t + b_2 t^2 \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$.

WHAT IS $\langle \underbrace{2 + 7t - 3t^2}_{\underline{p}} | \underbrace{1 + t^2}_{\underline{q}} \rangle$?

$\langle \underline{p} | \underline{q} \rangle = 2 + 0 - 3 = -1$.

EX. $V = C([0,1])$, $\langle f(x) | g(x) \rangle = \int_0^1 f(x) g(x)\, dx$.

INFINITE-DIM SPACE!

WHAT IS $\langle \underbrace{2 + 7t - 3t^2}_{\underline{f}} | \underbrace{1 + t^2}_{\underline{g}} \rangle$?

$$\langle \underline{f} | \underline{g} \rangle = \int_0^1 (2 + 7t - t^2 + 7t^3 - 3t^4)\, dt$$

$$= 2 + \frac{7}{2} - \frac{1}{3} + \frac{7}{4} - \frac{3}{5}.$$

Lecture 24
03/21/12

V inner product space, $\langle \cdot | \cdot \rangle : V \times V \to \mathbb{C}$.

Cauchy-Schwarz inequality: $|\langle \underline{x} | \underline{y} \rangle| \leq \|\underline{x}\| \, \|\underline{y}\|$.

Pf. (HW problem) (with $=$ iff $\underline{x}, \underline{y}$ linearly dependent).

Idea: (i) $0 \leq \|\underline{x} - t\underline{y}\|^2 = \langle \underline{x} - t\underline{y} | \underline{x} - t\underline{y} \rangle$

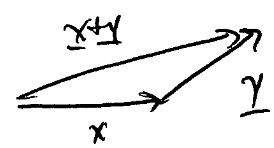
for any $t \in \mathbb{C}$.

(ii) assuming $\underline{y} \neq \underline{0}$ (otherwise ineq. is trivial),

take $t = \dfrac{\langle \underline{y} | \underline{x} \rangle}{\|\underline{y}\|^2}$. This choice of t

minimizes $\|\underline{x} - t\underline{y}\|^2$.

Corollary (triangle ineq.) $\|\underline{x} + \underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|$.



Pf. $\|\underline{x}+\underline{y}\|^2 = \langle \underline{x}+\underline{y} | \underline{x}+\underline{y} \rangle$

$$= \|\underline{x}\|^2 + \|\underline{y}\|^2 + \underbrace{\langle \underline{x} | \underline{y} \rangle + \langle \underline{y} | \underline{x} \rangle}$$

$$= 2 \operatorname{Re} \langle \underline{x} | \underline{y} \rangle$$

$$\leq 2 \, |\langle \underline{x} | \underline{y} \rangle|$$

by Cauchy-Schwarz $\leq 2 \, \|\underline{x}\| \, \|\underline{y}\|$

$$\leq (\|\underline{x}\| + \|\underline{y}\|)^2.$$

EX. FOR $V = \mathbb{C}^2$ WITH STD. INNER PRODUCT $\langle \underline{w} | \underline{z} \rangle = \overline{\underline{w}}^T \underline{z}$, VERIFY CAUCHY-SCHWARZ WITH $\underline{w} = (i, 1+i)^T$, $\underline{z} = (-i, 2)^T$.

$$\|\underline{w}\| = \sqrt{\langle (i, 1+i)^T | (i, 1+i)^T \rangle}$$
$$= \sqrt{(-i, 1-i)(i, 1+i)^T} = \sqrt{1 + (1+1)} = \sqrt{3}.$$

$$\|\underline{z}\| = \sqrt{\langle (-i, 2)^T | (-i, 2)^T \rangle}$$
$$= \sqrt{(i, 2)(-i, 2)^T} = \sqrt{1+4} = \sqrt{5}.$$

$$|\langle \underline{w} | \underline{z} \rangle| = |\langle (i, 1+i)^T | (-i, 2)^T \rangle|$$
$$= |(-i, 1-i)(-i, 2)^T| = |-1 + (2-2i)|$$
$$= |1 - 2i| = \sqrt{5}.$$

$$\Rightarrow |\langle \underline{w} | \underline{z} \rangle| \leq \|\underline{w}\| \|\underline{z}\|. \checkmark$$

DUALITY (6.3)

SUPPOSE $V = \mathbb{R}^n$ WITH STANDARD INNER PRODUCT $\langle \underline{x} | \underline{y} \rangle = \underline{x}^T \underline{y}$.

Q: HOW TO INTERPRET $\underbrace{\langle \underline{x} | \underline{y} \rangle}_{\text{"BRACKET"}}$?

A: $\langle \underline{x} | \underline{y} \rangle = \underline{x}^T \underline{y} \in \mathbb{R}$

COLUMN VECTOR $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$

ROW VECTOR

$(x_1, \ldots, x_n) \notin \mathbb{R}^n$!

SHOULD THINK OF $\langle \underline{x} | \underline{y} \rangle$ NOT AS A PRODUCT OF TWO VECTORS IN $\mathbb{R}^n$, BUT AS THE ACTION OF A LINEAR TRANSFORMATION (FUNCTIONAL) INDUCED BY $\underline{x} \in \mathbb{R}^n$ ON THE VECTOR $\underline{y} \in \mathbb{R}^n$!

GIVEN $\underline{x} \in \mathbb{R}^n$, DEFINE $L_{\underline{x}} : \mathbb{R}^n \longrightarrow \mathbb{R}$ BY

$$L_{\underline{x}}(\underline{y}) = \underline{x}^T \underline{y} \quad \text{FOR ALL } \underline{y} \in \mathbb{R}^n.$$

__FACT:__ __ANY__ LINEAR TRANSFORMATION L FROM $\mathbb{R}^n \longrightarrow \mathbb{R}$ IS OF THIS FORM FOR SOME $\underline{x} \in \mathbb{R}^n$!

(IN INFINITE-DIM. SPACES, THE ANALOGUE OF THIS IS KNOWN AS THE __REISZ REPRESENTATION THM.__)

__"PF."__

$$L(\underline{y}) = L\left(\sum_{i=1}^{n} a_i \underline{e}_i\right) = \sum_{i=1}^{n} a_i L(\underline{e}_i)$$

$$= \underbrace{\big(L(\underline{e}_1) \cdots L(\underline{e}_n)\big)}_{\text{CALL THIS } \underline{x}^T} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{= \underline{y}}.$$

$1 \times n$ MATRIX (ROW VECTOR)

__DIRAC NOTATION:__

$$\underbrace{\langle \underline{x} | \underline{y} \rangle}_{\text{"BRACKET"}} = \underbrace{\langle \underline{x} |}_{\substack{\text{"BRA"} \\ \parallel \\ L_{\underline{x}}}} \big(\underbrace{| \underline{y} \rangle}_{\substack{\text{"KET"} \\ \parallel \\ \underline{y}}} \big)$$

$|\underline{y}\rangle \in V$

$\langle \underline{x} | \in V' =$ __DUAL SPACE__ OF V (SPACE OF ALL LINEAR FUNCTIONALS ON V).