M346 (55820), Sample Final Exam Solutions

1.

i. For which $z \in \mathbb{C}$ is the sequence $\boldsymbol{v} = (a_1, a_2, a_3, \ldots)$, $a_n = z^n$, in $l_2(\mathbb{C})$? Why?

Solution: Since $\|\boldsymbol{v}\|_{l_2(\mathbb{C})}^2 = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |z|^2$, the series converges if and only if $|z| < 1$ (geometric series).

ii. For which $p \geq 0$ is the sequence $\boldsymbol{v} = (a_1, a_2, a_3, \ldots)$, $a_n = (2 + n^p)^{-1}$, in $l_2(\mathbb{C})$? Why?

Solution: Since $\|\boldsymbol{v}\|_{l_2(\mathbb{C})}^2 = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} \left|\frac{1}{2+n^p}\right|^2$, the series converges if and only if $p > 1/2$ by the limit comparison test for infinite series.

2. Compute the Fourier sine series, standard real Fourier series, and standard complex Fourier series of the function $f(x) = \sin(\pi x)$ on the interval $[0, 1]$. [Hint: Use the trigonometric identity $2\sin(u)\cos(v) = \sin(u+v) + \sin(u-v)$, if needed.]

Solution: $c_1 = 1$ and $c_n = 0$ for $n \geq 2$ (Fourier sine series); $\alpha_n = \frac{-4}{(4n^2-1)\pi}$, $\beta_n = 0$ for for $n \geq 0$ (standard real Fourier series); $\gamma_n = \frac{-2}{(4n^2-1)\pi}$ for $n \in \mathbb{Z}$ (standard complex Fourier series).

3. Using Fourier sine series, find the solution $u(x, t)$ to the time-dependent Schrodinger equation for a free particle in a 1-dimensional box:

$$\begin{cases} i\partial_t u = -\partial_{xx} u \\ u(0,t) = 0, u(a,t) = 0 \\ u(x,0) \text{ given} \end{cases}, \qquad x \in [0, a],\ t \geq 0.$$

(Here, $i = \sqrt{-1}$ is the imaginary constant.) That is, find the Fourier coefficients of the solution in terms of the Fourier coefficients of the initial data $u(x,\, 0)$. Are the modes of the system stable, neutrally stable, or unstable? How does the solution behave and how does this differ from the heat equation studied earlier?

Solution: The solution is $u(x,\, t) = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi x}{a}\right)$ with $c_n(t) = e^{i\lambda_n t} c_n(0)$, where $\lambda_n = -\frac{n^2\pi^2}{a^2}$ and $\{c_n(0)\}_{n=1}^{\infty}$ are the Fourier coefficients of the initial data $u(x,\, 0)$. We therefore see that the modes $\left\{\sin\left(\frac{n\pi x}{a}\right)\right\}_{n=1}^{\infty}$ of the system are all neutrally stable since $\mathrm{Re}(i\lambda_n) = 0$ for all n. Using Euler's formula, we see that the solution takes the form

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \sin\left(\frac{n^2\pi^2 t}{a^2}\right) \sin\left(\frac{n\pi x}{a}\right) + b_n \cos\left(\frac{n^2\pi^2 t}{a^2}\right) \sin\left(\frac{n\pi x}{a}\right) \right\}$$

for some set of complex-valued constants $\{a_n,\, b_n\}_{n=1}^{\infty}$ which describes a wave in space and time (called a plane wave). This is significantly different from the behavior of the heat equation, where all modes of the system decayed and the solution converges to 0 everywhere as $t \to \infty$.