M346 (55820), Sample Midterm #2 Questions

Below are some sample midterm questions. Please note that the intent of these is to help prepare for the exam, and that actual exam questions will not merely be modifications of these problems. Consult your HW, lecture notes, and books for additional sources of material to review. Finally, to obtain the full effect of an exam please complete these problems under time-pressure (50 minutes or less).

1. Let $A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$.

   a) Find the eigenvalues of A.

   b) Verify that you have obtained the correct eigenvalues by using the trace of A. Compute the determinant of A using the eigenvalues.

   c) Is A diagonalizable? Why or why not?

   d) Write A in its Jordan normal form $P\tilde{D}P^{-1}$ for an appropriate $\tilde{D}$ and P.

2. The matrix $A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ has eigenvalues -2, 1, and 1 with corresponding eigenvectors $\boldsymbol{b}_1 = (1,1,1)^T$, $\boldsymbol{b}_2 = (1,-1,0)^T$, and $\boldsymbol{b}_3 = (1,0,-1)^T$.

   a) Suppose $\boldsymbol{x}(k) = A\boldsymbol{x}(k-1)$ for all $k \geq 1$ with initial condition $\boldsymbol{x}(0) = (6,-1,-2)^T = \boldsymbol{b}_1 + 2\boldsymbol{b}_2 + 3\boldsymbol{b}_3$. Compute the solution $\boldsymbol{x}(k) = (x_1(k), x_2(k), x_3(k))^T$ explicitly.

   b) For part (a), what are the stable, unstable, and neutrally stable modes? Determine the limit of the ratios $x_1(k)/x_2(k)$ and $x_1(k)/x_3(k)$ as $k \to \infty$.

   c) Now consider the continuous-time system $d\boldsymbol{x}(t)/dt = A\boldsymbol{x}(t)$ for $t \geq 0$ with initial condition $\boldsymbol{x}(0) = (6,-1,-2)^T$. Compute the solution $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t))^T$ explicitly.

   d) For part (c), what are the stable, unstable, and neutrally stable modes? Determine the limit of the ratios $x_1(t)/x_2(t)$ and $x_1(t)/x_3(t)$ as $t \to \infty$.

3. Suppose $A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

   a) Diagonalize the matrix by writing it as $A = PDP^{-1}$.

   b) Write the matrix exponential e^D as $E + iF$, where E and F are real matrices.

4. Let $T = \begin{pmatrix} 0.8 & 0.4 & 0 & 0 \\ 0.2 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0.2 \\ 0 & 0 & 0.4 & 0.8 \end{pmatrix}$.

   a) Is T a transition matrix? If so, draw the states of the Markov chain with directed edges between states and their corresponding transition probabilities.

b) Is T irreducible? Is it aperiodic? Is it a regular transition matrix?

c) Determine all possible stationary distributions $\boldsymbol{\pi}$ of the Markov chain.

d) Suppose $S = 0.8T + 0.2B$, where B is a matrix with all entries $1/4$. Estimate the rate of convergence of $\boldsymbol{x}(k) = S\boldsymbol{x}(0)$ to the unique stationary distribution $\boldsymbol{\pi}_S$ as $k \to \infty$.

5. True or false? Explain your answer by providing a complete justification if true, and a counterexample if false.

a) If A and B are similar (i.e., $B = PAP^{-1}$) then they have the same spectrum.

b) If A has eigenvalues $(1 \pm i)/2$, any nonzero solution to the discrete-time system $\boldsymbol{x}(k) = A\boldsymbol{x}(k-1)$ will have oscillations that grow arbitrarily large in magnitude for k large.

c) If A has eigenvalues $-1 \pm 4i$, any nonzero solution to the continuous-time system $d\boldsymbol{x}/dt = A\boldsymbol{x}$ will have oscillations with frequency 4 and amplitudes that decay exponentially in time.

d) For regular transition matrices A, every column of A^k converges to the stationary distribution $\boldsymbol{\pi}$ as $k \to \infty$.

e) The total number of cycles of length r in a directed network is $(\mathrm{Tr}(A))^r$.