M346 (55820), Sample Midterm #3 Solutions

1.

a) Consider $\mathbb{R}^3$ with the standard inner product. Convert the basis $\mathcal{B} = \{(1,2,0)^T, (3,1,1)^T, (4,3,-5)^T\}$ into an orthonormal basis.

Solution: Using Gram-Schmidt, an orthonormal basis is $\mathcal{E} = \left\{\frac{1}{\sqrt{5}}(1,2,0)^T, \frac{1}{\sqrt{6}}(2,-1,1)^T, \frac{1}{\sqrt{30}}(2,\,-1,\,-5)^T\right\}$. Your answer may be different if the order of vectors in your orthogonalization procedure is different from the obvious one.

b) Find the matrix of the projection P_W onto the subspace $W = \operatorname{span}\{(1,2,0)^T, (3,1,1)^T\}$. Use this to compute $P_{W^\perp}\boldsymbol{v}$, where $\boldsymbol{v} = (1,2,3)^T$, where $W^\perp$ is the orthogonal complement of W (the subspace of all vectors orthogonal to W).

Solution: $P_W = P_{\boldsymbol{e}_1} + P_{\boldsymbol{e}_2} = |\boldsymbol{e}_1\rangle\langle\boldsymbol{e}_1| + |\boldsymbol{e}_2\rangle\langle\boldsymbol{e}_2| = \frac{1}{5}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$.

c) On $\mathbb{R}_2[t]$ with inner product $\langle p|q\rangle = \int_0^2 p(t)q(t)dt$, transform $\{1,\, t,\, t^2\}$ into an orthogonal basis (does not need to be orthonormal).

Solution: $\mathcal{D} = \{1, t-1, t^2 - 2t + 2/3\}$.

2.

a) Find the equation of the best line through the points $(1,-4)$, $(2,1)$, and $(3,2)$. Is this line unique?

Solution: Fitting the model $y = c + dx$ we have that $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{b} = (-4,1,2)^T$, so $A^*A = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$ and $A^*\boldsymbol{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$. Solving the normal equation $A^*A\boldsymbol{x}_{\text{LS}} = A^*\boldsymbol{b}$ gives the unique least-squares solution $\boldsymbol{x}_{\text{LS}} = (-19/3, 3)^T$ so the best line is $y = -19/3 + 3x$.

b) Let W be the subspace of $\mathbb{R}^3$ spanned by $(1,\,2,\,3)^T$ and $(1,\,1,\,1)^T$. Find the point in W which lies closest to $(-4,1,2)^T$. Justify your answer.

Solution: The closest point to $\boldsymbol{b}$ which lies in $\operatorname{Ran}(A)$ is $A\boldsymbol{x}_{\text{LS}} = (-10/3, -1/3, 8/3)^T$.

3. Let $A = \begin{pmatrix} 4 & 2 & -2 & 2 \\ 3 & -1 & 2 & -3 \end{pmatrix}$.

a) What is the rank r of A?

Solution: $r = 2$.

b) Write the singular value decomposition (SVD) of A as a sum of r terms (you do not need to expand your answers as a matrix). [Hint: Remember that the eigenvalues and eigenvectors of A^*A and AA^* are intimately related! Choose the easiest matrix to work with.]

Solution: We work with AA^* since this is a smaller matrix than A^*A. The eigenvalues of AA^* are $\sigma_1 = 2\sqrt{7}$ and $\sigma_2 = \sqrt{23}$, with corresponding orthonormal eigenvectors $\boldsymbol{u}_1 = (1, 0)^T$ and $\boldsymbol{u}_2 = (0,1)^T$. Then A^*A has the same eigenvalues with corresponding eigenvectors $\boldsymbol{v}_1 = \frac{1}{\sigma_1}A^*\boldsymbol{u}_1 = \frac{1}{\sqrt{7}}(2, 1, -1, 1)^T$ and $\boldsymbol{v}_2 = \frac{1}{\sigma_2}A^*\boldsymbol{u}_2 = \frac{1}{\sqrt{23}}(3, -1, 2, 3)^T$. So the SVD of A is $A = \sum_{i=1}^2 \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^*$.

c) Compute the error between A and its best rank-one approximation.

Solution: Since the best rank-one approximation is $A_1 = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^*$, the approximation error is $\|A - A_1\| = \sqrt{\sigma_2^2} = \sqrt{23}$ in the Frobenius norm.

4. Consider the symmetric matrix $A = \begin{pmatrix} 24 & 7 \\ 7 & -24 \end{pmatrix}$.

a) Write $A = UDU^*$ for an appropriate diagonal matrix D and unitary matrix U.

Solution: $D = \begin{pmatrix} 25 & 0 \\ 0 & -25 \end{pmatrix}$, $U = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & 1 \\ 1 & -7 \end{pmatrix}$.

b) Express $\boldsymbol{x} = (13, 9)^T$ as a linear combination of the eigenvectors found in part (a).

Solution: $\boldsymbol{x} = 5\sqrt{2}(2\boldsymbol{u}_1 - \boldsymbol{u}_2)$ where $\boldsymbol{u}_1, \boldsymbol{u}_2$ are the columns of U.

c) Let $|A| = U|D|U^*$, where $|D|$ is the diagonal matrix of *magnitudes* of the eigenvalues of A. Show that $|A|$ is positive and compute $\sqrt{|A|}$.

Solution: $|A| = U|D|U^*$ with $|D| = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$. It is easy to see that $|A|$ is self adjoint and has nonnegative eigenvalues, and is therefore positive. Then we have that $\sqrt{|A|} = U|D|^{1/2}U^* = \frac{1}{50}\begin{pmatrix} 7 & 1 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 1 & -7 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$.

5. True or false? Justify your answers.

a) The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ has orthogonal eigenvectors.

Solution: True. This holds by the spectral theorem since the matrix is normal.

b) $\frac{1}{\sqrt{7}}\begin{pmatrix} 2-i & -1+i \\ 1+i & 2+i \end{pmatrix}$ is unitary.

Solution: True. The columns of the matrix are orthonormal.

c) If a matrix $A \in M_{n,n}(\mathbb{C})$ satisfies $A = A^T$ then the eigenvalues of A are necessarily real.

Solution: False. If the entries are complex then this does not necessarily hold.

d) If $\langle f|g\rangle = \int_0^\infty f(x)g(x)e^{-x}dx$ for functions $f,\, g \in L_2([0,\infty))$ and $L = x + \frac{d}{dx}$ (assume that all elements of $L_2([0,\infty))$ are differentiable), its adjoint is $L^* = x - \frac{d}{dx}$.

Solution: False. Integration by parts shows that the adjoint is actually $L^* = (x+1) - \frac{d}{dx}$.

e) A real matrix A admits an SVD $A = U\Sigma V^*$ where U, V, Σ are all real matrices.

Solution: False. The eigenvalues of a real matrix may be complex.