LECTURE 03
01/23/12

SUPPOSE $\underline{v}_i \in \mathbb{R}^n$, $i=1,\dots,m$.

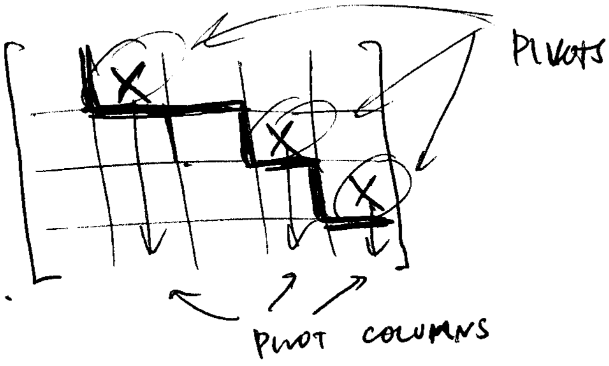
$$c_1\underline{v}_1+\cdots+c_m\underline{v}_m = n\Big\updownarrow\underbrace{\underset{\longleftrightarrow\ m}{\begin{pmatrix} \underline{v}_1 & \cdots & \underline{v}_m\end{pmatrix}}}_{\text{MATRIX}} \begin{pmatrix} c_1\\ \vdots \\ c_m\end{pmatrix}$$

MATRIX $A=(\underline{v}_1,\dots,\underline{v}_m)\in M_{n,m}$.

Q: HOW TO ANALYZE A?

ROW ECHELON FORM (ref):

- ALL 0's BELOW PIVOTS
- ALL ZERO ROWS AT BOTTOM.



REDUCED ROW ECHELON FORM (rref) IF, IN ADDITION

- ALL 0's ABOVE PIVOTS
- ALL PIVOTS ARE 1's.

CAN PUT A INTO rref USING GAUSSIAN ELIMINATION, GIVING A_{rref}. # OF PIVOTS $\doteq$ RANK(A).

Ex.

$$\begin{pmatrix}\boxed{1}&0&0&2&3\\0&0&1&6&2\\0&0&1&2&3\end{pmatrix}\begin{matrix}\\ \\ -R_2\end{matrix} \longrightarrow \begin{pmatrix}\boxed{1}&0&0&2&3\\0&0&\boxed{1}&6&2\\0&0&0&\boxed{-4}&1\end{pmatrix}\begin{matrix}+\frac{2}{4}R_3\\+\frac{6}{4}R_3\\ \times -\frac{1}{4}\end{matrix}$$

NOT ref

ref, NOT rref.

$$\rightarrow \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & \frac{7}{2} \\ 0 & 0 & \boxed{1} & 0 & \frac{7}{2} \\ 0 & 0 & 0 & \boxed{1} & -\frac{1}{4} \end{pmatrix}$$

rref.

- SOLUTIONS TO $A\underline{x} = \underline{0}$ SAME AS SOLN'S TO $A_{rref}\underline{x} = \underline{0}$. (ELEMENTARY ROW OPERATIONS LEAVE $\underline{0}$ INVARIANT).
- COLUMNS W/O PIVOTS GIVE FREE VARIABLES (HERE, x_2 AND x_5 SINCE NO PIVOTS IN COLUMNS 2,5)
- ROWS W/O PIVOTS ARE $(0, \dots, 0)$ AND LEAD TO INCONSISTENCY.
- PIVOT COLUMNS OF A (NOT OF A_{rref}!) SPAN COLUMN SPACE OF A.

__Q:__ HOW TO SOLVE $A\underline{x} = \underline{b}$ WITH $A, \underline{b}$ GIVEN?

__AUGMENTED MATRIX__: $\left(A \mid \underline{b} \right)$.

i) REDUCE TO rref TO GET $\left(A_{rref} \mid \underline{b}_{rref} \right)$

ii) SOLVE FOR $\underline{x}$ USING BACK SUBSTITUTION,

$$A_{rref}\,\underline{x} = \underline{b}_{rref} \Rightarrow \underline{x} \text{ SOLVES } A\underline{x} = \underline{b}.$$

__EX.__ A AS BEFORE, $\underline{b} = (1,2,3)^T$.

$$\rightsquigarrow \underline{b}_{rref} = \left(\frac{3}{2}, \frac{7}{2}, -\frac{1}{4}\right)^T \Rightarrow$$

x_5 = FREE

$x_4 = -\frac{1}{4} + \frac{1}{4}x_5$

$x_3 = \frac{7}{2} - \frac{7}{2}x_5$

x_2 = FREE

$x_1 = \frac{3}{2} - \frac{7}{2}x_5$.

EX. WHAT IF INSTEAD

$$\left(A_{rref} \mid \underline{b}_{rref}\right) = \left(\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & \boxed{1} & 6 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & \underline{\underline{\boxed{-\frac{1}{4}}}} \end{array}\right) \longleftarrow \text{NONZERO.} \qquad ?$$

NO PIVOT IN LAST ROW OF A_{rref}

$\Rightarrow$ NO SOLN. TO $A\underline{x} = \underline{b}$.

THM. [THM. 2.3–2.6 IN SADUN, SECTIONS 2.1–2.3 IN TREIL]

WITH $A = (\underline{v}_1, \ldots, \underline{v}_m) \in M_{n,m}$

(i) SOLN. TO $A\underline{x} = \underline{b}$ (IF IT EXISTS!) IS UNIQUE.

$\Leftrightarrow$ ONLY SOLN. TO $A\underline{x} = \underline{0}$ IS $\underline{x} = 0$.

$\Leftrightarrow$ PIVOT IN EVERY COLUMN OF A_{rref} (I.E., NO FREE VARIABLES)

$\Leftrightarrow$ $\{\underline{v}_1, \ldots, \underline{v}_m\}$ LINEARY INDEP. $(\Rightarrow m \leq n)$.

(ii) SOLN. TO $A\underline{x} = \underline{b}$ EXISTS FOR ANY $\underline{b} \in \mathbb{R}^n$

$\Leftrightarrow$ PIVOT IN EVERY ROW OF A_{rref}

$\Leftrightarrow$ $\{\underline{v}_1, \ldots, \underline{v}_m\}$ SPANS $\mathbb{R}^n$ $(\Rightarrow m \geq n)$

(iii) SOLN. TO $A\underline{x} = \underline{b}$ EXISTS AND IS UNIQUE FOR ANY $b \in \mathbb{R}^n$

$\Leftrightarrow$ PIVOT IN EVERY ROW AND COLUMN OF A_{rref}

$\Leftrightarrow$ $\{\underline{v}_1, \ldots, \underline{v}_m\}$ BASIS OF $\mathbb{R}^n$ $(\Rightarrow m = n)$.

IN ADDITION, (iii) EQUIVALENT TO

$\Longleftrightarrow$ A IS INVERTIBLE

$\Longleftrightarrow$ $\det(A) \neq 0$

$\Longleftrightarrow$ $A_{rref} = I$.

EX. DO COLUMNS OF $\underbrace{\begin{pmatrix} 1 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \\ 1 & -1 & 1 & 4 \end{pmatrix}}_{A}$ SPAN $\mathbb{R}^3$?

$$A \curvearrowright \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -1 & 1 & -1 \\ 0 & -2 & 0 & 1 \end{pmatrix} \curvearrowright \begin{pmatrix} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{-1} & 1 & -1 \\ 0 & 0 & \boxed{-2} & 3 \end{pmatrix}$$

<u>ref</u>. PIVOT IN EVERY ROW

$\Rightarrow$ $\boxed{\text{Yes.}}$.

LECTURE 04
01/25/12

REPRESENTATION OF VECTORS IN A BASIS (2.3):

$\mathcal{B} = \{\underline{b}_1, \ldots, \underline{b}_n\}$ BASIS OF V.

FOR ANY $\underline{v} \in V$, $\underline{v} = a_1 \underline{b}_1 + \cdots + a_n \underline{v}_n$ FOR SOME UNIQUE SET OF COEFFICIENTS $\{a_1, \ldots, a_n\}$.

Q: WHY UNIQUE?

A: IF $\underline{v} = \sum a_i \underline{b}_i = \sum c_i b_i$ FOR SOME $\{a_i\}_{i=1}^n$, $\{c_i\}_{i=1}^n$,

$$\sum (a_i - c_i) \underline{b}_i = \underline{0} \implies a_i = c_i \text{ FOR ALL } i = 1, \ldots, n$$

SINCE $\{\underline{b}_i\}$ LINEARLY INDEP.

Def. $$[\underline{v}]_{\mathcal{B}} = \underbrace{(a_1, \ldots, a_n)}_{\text{COORDINATES OF } \underline{v} \text{ IN BASIS } \mathcal{B}.}{}^T \in \mathbb{R}^n$$

THM.
- $\underline{x}, \underline{y} \in V$, c SCALAR $\implies$ $[\underline{x}+\underline{y}]_{\mathcal{B}} = [\underline{x}]_{\mathcal{B}} + [\underline{y}]_{\mathcal{B}}$, $[c\underline{x}]_{\mathcal{B}} = c[\underline{x}]_{\mathcal{B}}$.
- $\underline{w}$ LINEAR COMBINATION OF $\{\underline{v}_1, \ldots, \underline{v}_m\}$ $\iff$ $[\underline{w}]_{\mathcal{B}}$ LINEAR COMBINATION OF $\{[\underline{v}_1]_{\mathcal{B}}, \ldots, [\underline{v}_m]_{\mathcal{B}}\}$.
- $\{\underline{v}_i\}_{i=1}^m$ LINEARLY INDEP. $\iff$ $\{[v_i]_{\mathcal{B}}\}_{i=1}^m$ LINEARLY INDEP.

Pf. EASY.

THIS IMPLIES THAT $[\cdot]_{\mathcal{B}}$ IS AN <u>ISOMORPHISM</u> FROM V TO $\mathbb{R}^n$, AND SO, V IS <u>ISOMORPHIC</u> TO $\mathbb{R}^n$. ($V \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^n$, $\underline{b}_i \xrightarrow{[\cdot]_{\mathcal{B}}} \underline{e}_i$, $\underline{v} = \sum a_i b_i \xrightarrow{[\cdot]_{\mathcal{B}}} [v]_{\mathcal{B}} = \sum a_i e_i$)

<u>EX.</u> CONSIDER SUBSPACE $\tilde{\mathbb{R}}_3[t]$ OF $\mathbb{R}_3[t]$ CONSISTING OF POLYNOMIALS WITH $p(0) = 0$. ARE

$$p_1 = t - t^3, \quad p_2 = t^2 + 3t^3, \quad p_3 = 2t + 3t^2 + 4t^3$$

a) LINEARLY INDEP.?

b) SPAN $\tilde{\mathbb{R}}_3[t]$?

c) BASIS OF $\tilde{\mathbb{R}}_3[t]$?

<u>Ans:</u> USE A BASIS $\mathcal{B}$ OF $\tilde{\mathbb{R}}_3[t]$ TO TRANSFORM $\tilde{\mathbb{R}}_3[t]$ TO $\mathbb{R}^n$ FOR SOME APPROPRIATE n.

<u>STEP 1.</u> FIND A BASIS OF $\tilde{\mathbb{R}}_3[t]$:

ANY ELEMENT IN $\tilde{\mathbb{R}}_3[t]$ HAS FORM $p(t) = \overset{0}{\cancel{a_0}} + a_1 t + a_2 t^2 + a_3 t^3$

$\Rightarrow \mathcal{B} = \{t, t^2, t^3\}$ IS A BASIS.

<u>STEP 2.</u> WRITE p_1, p_2, p_3 IN COORDINATES OF $\mathcal{B}$:

$$[p_1]_{\mathcal{B}} = (1, 0, -1)^T, \quad [p_2]_{\mathcal{B}} = (0, 1, 3)^T, \quad [p_3]_{\mathcal{B}} = (2, 3, 4)^T.$$

<u>STEP 3.</u> ANALYZE MATRIX $\underbrace{([p_1]_{\mathcal{B}} \cdots [p_3]_{\mathcal{B}})}_{A.}$:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -1 & 3 & 4 \end{pmatrix}_{+R_1} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 3 & 6 \end{pmatrix}_{-3R_2}$$

$$\rightarrow \begin{pmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{-3} \end{pmatrix}$$

<u>ref.</u> PIVOT IN EVERY ROW, COLUMN.

$\Longleftrightarrow \{[p_i]_{\mathcal{B}}\}_{i=1}^{3}$ BASIS OF $\mathbb{R}^3$

$\Longleftrightarrow \{p_i\}_{i=1}^{3}$ BASIS OF $\tilde{\mathbb{R}}_3[t]$ (BY THM.).

<u>**DEF.**</u> IF V ~~HAS~~ BASIS $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ OF n VECTORS, SAY $\dim(V)$ IS n (I.E, V IS n-DIMENSIONAL).

DIMENSION IS INDEPENDENT OF CHOICE OF BASIS!

WHY? IF $\mathcal{D} = \{\underline{d}_1, \dots, \underline{d}_m\} \subset V$,

$m < n \Rightarrow \mathcal{D}$ DOES NOT SPAN V

$m > n \Rightarrow \mathcal{D}$ LINEARLY DEPENDENT

SO, $\mathcal{D}$ BASIS $\Longrightarrow m = n$.

CHANGE OF BASIS (2.3):

$\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ BASES OF V

$\mathcal{D} = \{\underline{d}_1, \dots, \underline{d}_n\}$

FIX $\underline{v} \in V$. HOW TO RELATE $[\underline{v}]_{\mathcal{B}}$ TO $[\underline{v}]_{\mathcal{D}}$?

CLAIM: THERE IS A UNIQUE $n \times n$ MATRIX $P_{\underset{\text{TO HERE}}{\mathcal{D}}\underset{\text{FROM HERE}}{\mathcal{B}}}$

SUCH THAT $[\underline{v}]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}$

FOR ALL $\underline{v} \in V$.

PF. $\underline{v} = \sum_{i=1}^{n} a_i \underline{b}_i \Rightarrow [\underline{v}]_{\mathcal{D}} = \left[\sum_{i=1}^{n} a_i \underline{b}_i\right]_{\mathcal{D}}$

$$= \sum_{i=1}^{n} a_i [\underline{b}_i]_{\mathcal{D}}$$

$$= \underbrace{\left([\underline{b}_1]_{\mathcal{D}} \cdots [\underline{b}_n]_{\mathcal{D}}\right)}_{n \times n \text{ MATRIX}} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{\overset{\shortparallel}{[\underline{v}]_{\mathcal{B}}}}$$

$$= P_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}.$$

TO CHECK THAT $P_{\mathcal{D}\mathcal{B}} = \left([\underline{b}_1]_{\mathcal{D}} \cdots [\underline{b}_n]_{\mathcal{D}}\right)$ IS

THE ONLY MATRIX WHICH SATISFIES $[\underline{v}]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}$,

NOTE THAT IF $[\underline{v}]_{\mathcal{D}} = A [\underline{v}]_{\mathcal{B}}$ FOR ALL $\underline{v} \in V$,

THEN $[\underline{b}_i]_{\mathcal{D}} = A [\underline{b}_i]_{\mathcal{B}} = A e_i$ = ith COLUMN OF A. ✓

PICTURE:

$$\begin{tikzcd}
 & \mathbb{R}^n \arrow[dd, shift left, "P_{\mathcal{D}\mathcal{B}}"] \\
V \arrow[ur, "{[\cdot]_{\mathcal{B}}}"] \arrow[dr, "{[\cdot]_{\mathcal{D}}}"'] & \\
 & \mathbb{R}^n \arrow[uu, shift left] \arrow[uu, bend right=80, "P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}^{-1}"']
\end{tikzcd}$$

THM. $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}^{-1}$.

PF. WANT $P_{\mathcal{B}\mathcal{D}} P_{\mathcal{D}\mathcal{B}} = I$. LET $\underline{y} \in \mathbb{R}^n$. THERE IS SOME $\underline{x} \in V$ S.T. $[\underline{x}]_{\mathcal{B}} = \underline{y}$. SO,

$$P_{\mathcal{B}\mathcal{D}} P_{\mathcal{D}\mathcal{B}} \underline{y} = P_{\mathcal{B}\mathcal{D}} \underbrace{P_{\mathcal{D}\mathcal{B}} [\underline{x}]_{\mathcal{B}}}_{[\underline{x}]_{\mathcal{D}}} = P_{\mathcal{B}\mathcal{D}} [\underline{x}]_{\mathcal{D}} = [\underline{x}]_{\mathcal{B}} = \underline{y}.$$

$$\Rightarrow \quad P_{\mathcal{B}\mathcal{D}} P_{\mathcal{D}\mathcal{B}} = I.$$

EX. $M_{2,2}$ = 2×2 MATRICES W/ REAL ENTRIES.

$$\mathcal{E} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

STANDARD BASIS.

SUPPOSE $\underline{v} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

a) WHAT IS $[\underline{v}]_{\mathcal{E}}$?

$$[\underline{v}]_{\mathcal{E}} = (1,2,4,3)^T.$$

b) IF $\mathcal{D} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

ANOTHER BASIS OF $M_{2,2}$, WHAT IS $[\underline{v}]_{\mathcal{D}}$?

STEP 1. COMPUTE $P_{\mathcal{E}\mathcal{D}} = \left([\underline{d}_1]_{\mathcal{E}} \cdots [\underline{d}_4]_{\mathcal{E}} \right)$

SINCE THIS IS EASY TO COMPUTE VERSUS $P_{\mathcal{D}\mathcal{E}}$.

THEN, $P_{\mathcal{E}\mathcal{D}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix}$.

STEP 2. INVERT $P_{\mathcal{E}\mathcal{D}}$ TO GET $P_{\mathcal{D}\mathcal{E}}$.

USUALLY, DO THIS BY COMPUTER. IN THIS CASE, EASY TO DO BY HAND TO GET THAT

$$P_{\mathcal{D}\mathcal{E}} = P_{\mathcal{E}\mathcal{D}}^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}.$$

STEP 3.

$$[\underline{v}]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{E}} [\underline{v}]_{\mathcal{E}} = (2, -1, 3, -1)^T.$$

EASY TO CHECK IN THIS EX. THAT THIS IS CORRECT!

$$2\underline{d}_1 - \underline{d}_2 + 3\underline{d}_3 - \underline{d}_4 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \underline{v} \ \checkmark.$$

LECTURE 05
01/27/12

LINEAR TRANSFORMATIONS, OPERATORS (3.1):

DEF. $L : V \to W$ LINEAR TRANSFORMATION (LINEAR OPERATOR IF $W = V$, LINEAR FUNCTIONAL IF $W = \mathbb{R}^n$) IF FOR EVERY $\underline{v}_1, \underline{v}_2 \in V$, SCALAR c,

$$\begin{cases} L(\underline{v}_1 + \underline{v}_2) = L(\underline{v}_1) + L(\underline{v}_2) \\ L(c\underline{v}_1) = c\,L(\underline{v}_1) \end{cases}$$

EX. $V = W = \mathbb{R}$.

$L(x) = |x|$ IS <u>NOT</u> A LINEAR TRANSFORMATION SINCE $L(-x) = |-x| = |x| = L(x) \neq -L(x)$.

EX. $V = \mathbb{R}^n$, $W = \mathbb{R}^m$

$L(\underline{v}) = A\underline{v}$ FOR $A \in M_{m,n}$.

IN FACT, <u>EVERY</u> LINEAR TRANSFORMATION FROM $\mathbb{R}^n$ TO $\mathbb{R}^m$ HAS THIS FORM. WHY?

IF $\underline{v} = (a_1, \dots, a_n) \in \mathbb{R}^n$,

$$L(\underline{v}) = L\left(\sum_{i=1}^n a_i \underline{e}_i\right) = \sum_{i=1}^n a_i\, L(\underline{e}_i) = \overbrace{\begin{pmatrix} L(\underline{e}_1) & \cdots & L(\underline{e}_n) \end{pmatrix}}^{m\times n \text{ MATRIX } A} \overset{``\underline{v}"}{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}$$

EX. V n-DIM. VECTOR SPACE WITH BASIS $\mathcal{B}$,

$W = \mathbb{R}^n$

$L(\underline{v}) = [\underline{v}]_{\mathcal{B}}$ LINEAR TRANSFORMATION.

EX. $V = \mathbb{R}_n[t]$, $W = \mathbb{R}_m[t]$

$L(p(t)) = \frac{dp}{dt}(t)$ IS A LINEAR TRANSFORMATION SO LONG AS $m \geq n-1$ (OTHERWISE, $L : V \not\to W$!).

EX. $V = C^\infty[0,1]$

$\nwarrow$ SPACE OF INFINITELY DIFFERENTIABLE FCN.'S. ON $[0,1]$.

$L : V \to V$ IS A LINEAR OPERATOR WHEN

$L(f(x)) = \frac{df}{dx}(x)$ (NOTATION: $L = \frac{d}{dx}$).

EX. $V = C^0[0,1]$, $W = \mathbb{R}$.

$L(f) = \int_0^1 f(s)\, ds$ IS A LINEAR FUNCTIONAL.

MATRIX REPRESENTATION OF LINEAR TRANSFORMATION (3.2)

Q: DO WE NEED TO SPECIFY $L(\underline{v})$ FOR EVERY $\underline{v} \in V$?

A: ONLY NEED $\{L(\underline{b}_i)\}_{i=1}^n$ FOR BASIS $\mathcal{B} = \{\underline{b}_i\}_{i=1}^n$ OF V, SINCE

$$L(\underline{v}) = L\left(\sum_{i=1}^n a_i \underline{b}_i\right) = \sum_{i=1}^n a_i L(\underline{b}_i).$$

Q: CAN WE WRITE THIS IN MATRIX FORM ?

A: ONLY AFTER WE'VE CHOSEN A BASIS $\mathcal{D} = \{\underline{d}_1, \dots, \underline{d}_m\}$ FOR W AND WRITTEN EVERYTHING IN COORDINATES !

$$\begin{aligned}
[L(\underline{v})]_{\mathcal{D}} &= \left[\sum_{i=1}^n a_i L(\underline{b}_i)\right]_{\mathcal{D}} \\
&= \sum_{i=1}^n a_i [L(\underline{b}_i)]_{\mathcal{D}} \\
&= \underbrace{\left([L(\underline{b}_1)]_{\mathcal{D}} \cdots [L(\underline{b}_n)]_{\mathcal{D}}\right)}_{m \times n \text{ MATRIX } [L]_{\mathcal{D}\mathcal{B}}} \underbrace{\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}}_{[\underline{v}]_{\mathcal{B}}} \\
&= [L]_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}.
\end{aligned}$$

THIS MATRIX $[L]_{\mathcal{D}\mathcal{B}}$ DEPENDS ON CHOICE OF BASES $\mathcal{B}$ OF V AND $\mathcal{D}$ OF W !!

IN FACT, IT IS THE UNIQUE MATRIX REPRESENTATION W/ RESPECT TO THIS CHOICE OF BASES.

PICTURE:

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & W \arrow[d, "{[\cdot]_{\mathcal{D}}}"] \\ \mathbb{R}^n \arrow[r, "{[L]_{\mathcal{D}\mathcal{B}}}"'] & \mathbb{R}^m \end{tikzcd}$$

REMARKS:

(i) IF $V=W$, ONLY NEED ONE BASIS $\mathcal{B}$. IN THIS CASE, DENOTE $[L]_{\mathcal{B}\mathcal{B}}$ BY $[L]_{\mathcal{B}}$.

(ii) MOST OF THIS COURSE FOCUSES ON CHOICE OF $\mathcal{B}$ AND $\mathcal{D}$ SO THAT $[L]_{\mathcal{D}\mathcal{B}}$ LOOKS SIMPLE (I.E., IS DIAGONAL!).