LECTURE 06
01/30/12

LAST TIME: $$[L(\underline{v})]_{\mathcal{D}} = [L]_{\mathcal{D}\mathcal{B}} [\underline{v}]_{\mathcal{B}}$$

UNIQUE MATRIX REPRESENTATION OF L IN BASIS $\mathcal{B}$ OF V, $\mathcal{D}$ OF W.

$$[L]_{\mathcal{D}\mathcal{B}} \doteq \left([L(\underline{b}_1)]_{\mathcal{D}} \cdots [L(\underline{b}_n)]_{\mathcal{D}} \right).$$

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"'] & W \arrow[d, "{[\cdot]_{\mathcal{D}}}"] \\ \mathbb{R}^n \arrow[r, "{[L]_{\mathcal{D}\mathcal{B}}}"'] & \mathbb{R}^m \end{tikzcd}$$

- IF $V=W$, ONLY NEED ONE BASIS FOR BOTH SPACES, SAY $\mathcal{B}$. WILL DENOTE $[L]_{\mathcal{B}} \doteq [L]_{\mathcal{B}\mathcal{B}}$.
- MUCH OF THIS COURSE FOCUSES ON CHOICE OF BASIS $\mathcal{B}$ S.T. $[L]_{\mathcal{B}}$ LOOKS AS SIMPLE AS POSSIBLE (I.E, IS DIAGONAL!).

EX. $V = W = \mathbb{R}_2[t]$, BASIS $\mathcal{B} = \{1, t, t^2\}$.

(i) $L(p) = \frac{dp}{dt}$ (I.E., $L = \frac{d}{dt}$) IS LINEAR OPERATOR.

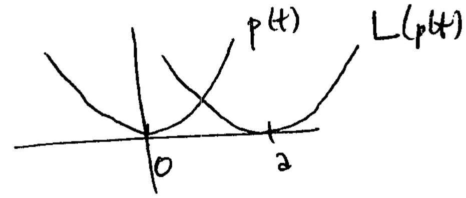
(ii) $$[L]_{\mathcal{B}} = \left([L(\underline{b}_1)]_{\mathcal{B}} \quad [L(\underline{b}_2)]_{\mathcal{B}} \quad [L(\underline{b}_3)]_{\mathcal{B}} \right)$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

SINCE
$$L(\underline{b}_1) = 0 \Rightarrow [L(\underline{b}_1)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$L(\underline{b}_2) = 1 \Rightarrow [L(\underline{b}_2)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$L(\underline{b}_3) = 2t \Rightarrow [L(\underline{b}_3)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}.$$

<u>Ex.</u> $V = W = \mathbb{R}_2[t]$, BASIS $\mathcal{B} = \{1, t, t^2\}$.

(i) $L(p(t)) = p(t-a)$ IS A LINEAR OPERATOR, CALLED THE <u>SHIFT OPERATOR</u> BY $a \in \mathbb{R}$.



(ii) $$[L]_{\mathcal{B}} = \left([L(\underline{b}_1(t))]_{\mathcal{B}} \;\cdots\; [L(\underline{b}_3(t))]_{\mathcal{B}} \right)$$

$$= \begin{pmatrix} 1 & -a & a^2 \\ 0 & 1 & -2a \\ 0 & 0 & 1 \end{pmatrix}$$ SINCE

$$L(\underline{b}_1(t)) = 1 \Rightarrow [L(\underline{b}_1(t))]_{\mathcal{B}} = (1,0,0)^T$$
$$L(\underline{b}_2(t)) = t-a \Rightarrow [L(\underline{b}_2(t))]_{\mathcal{B}} = (-a,1,0)^T$$
$$L(\underline{b}_3(t)) = (t-a)^2 = t^2 - 2at + a^2 \Rightarrow [L(\underline{b}_3(t))]_{\mathcal{B}} = (a^2, -2a, 1)^T$$

(ii) WHAT IS $[L]_{\mathcal{D}\mathcal{B}}$, WHERE $\mathcal{D} = \{1, t-a, (t-a)^2\}$ IS ANOTHER BASIS OF $\mathbb{R}_2[t]$?

$$[L]_{\mathcal{D}\mathcal{B}} = \left([L(\underline{b}_1)]_{\mathcal{D}} \;\cdots\; [L(\underline{b}_3)]_{\mathcal{D}} \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ SINCE
$$[L(\underline{b}_1(t))]_{\mathcal{D}} = (1,0,0)^T$$
$$[L(\underline{b}_2(t))]_{\mathcal{D}} = (0,1,0)^T$$
$$[L(\underline{b}_3(t))]_{\mathcal{D}} = (0,0,1)^T.$$

<u>NOTE:</u> CHOICE OF BASIS $\mathcal{B}$ IN DOMAIN V OF L AND BASIS $\mathcal{D}$ IN IMAGE SPACE W OF L GIVES DIFFERENT REPRESENTATIONS $[L]_{\mathcal{D}\mathcal{B}}$!

Effect of a change of basis (3.3):

Q: $L : V \longrightarrow W$

$\mathcal{B}$ basis of V
$\mathcal{B}'$ basis of W. $\Rightarrow$ $[L]_{\mathcal{B}'\mathcal{B}}$ representation of L.

What if we switch to new bases $\mathcal{D}$ of V and $\mathcal{D}'$ of W? How does $[L]_{\mathcal{D}'\mathcal{D}}$ relate to $[L]_{\mathcal{B}'\mathcal{B}}$?

A:

$$\begin{tikzcd}
 & V \arrow[r, "L"] \arrow[d, "{[\cdot]_{\mathcal{B}}}"] \arrow[dl, bend right, "{[\cdot]_{\mathcal{D}}}"'] & W \arrow[d, "{[\cdot]_{\mathcal{B}'}}"'] \arrow[dr, bend left, "{[\cdot]_{\mathcal{D}'}}"] & \\
\mathbb{R}^n \arrow[r, "P_{\mathcal{B}\mathcal{D}}"'] & \mathbb{R}^n \arrow[r, "{[L]_{\mathcal{B}'\mathcal{B}}}"'] & \mathbb{R}^m \arrow[r, "P_{\mathcal{D}'\mathcal{B}'}"'] & \mathbb{R}^m
\end{tikzcd}$$

Therefore (since we multiply matrices from right to left),

$$[L]_{\mathcal{D}'\mathcal{D}} = P_{\mathcal{D}'\mathcal{B}'}\,[L]_{\mathcal{B}'\mathcal{B}}\,P_{\mathcal{B}\mathcal{D}}$$

where $P_{\mathcal{B}\mathcal{D}}$ and $P_{\mathcal{D}'\mathcal{B}'}$ are change of basis matrices.

- If $V=W$ and we use common basis $\mathcal{B}$, matrix representation of $L : V \longrightarrow V$ in new basis $\mathcal{D}$ is

$$[L]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}\,[L]_{\mathcal{B}}\,P_{\mathcal{B}\mathcal{D}}$$

inverses of each other since $P_{\mathcal{B}\mathcal{D}} = P_{\mathcal{D}\mathcal{B}}^{-1}$.

EX. SUPPOSE $V = W$ HAS BASIS $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$.

LET $L : V \to V$ S.T. $L(\underline{b}_1) = 2\underline{b}_1 + \underline{b}_2$

$L(\underline{b}_2) = \underline{b}_1 + 2\underline{b}_2$.

(i) $[L]_{\mathcal{B}} = \left([L(\underline{b}_1)]_{\mathcal{B}} \quad [L(\underline{b}_2)]_{\mathcal{B}} \right) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(ii) IF $\mathcal{D} = \{ \underbrace{\underline{b}_1 + \underline{b}_2}_{\underline{d}_1}, \underbrace{\underline{b}_1 - \underline{b}_2}_{\underline{d}_2} \}$, WHAT IS $[L]_{\mathcal{D}}$?

$$P_{\mathcal{B}\mathcal{D}} = \left([\underline{d}_1]_{\mathcal{B}} \quad [\underline{d}_2]_{\mathcal{B}} \right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$P_{\mathcal{D}\mathcal{B}} = P_{\mathcal{B}\mathcal{D}}^{-1} = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

$$\Rightarrow \quad [L]_{\mathcal{D}} = P_{\mathcal{D}\mathcal{B}} [L]_{\mathcal{B}} P_{\mathcal{B}\mathcal{D}} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

NOTE: THIS IS EXACTLY THE DECOUPLING OF SECTION 1!

FOR EX., $\frac{d\underline{x}}{dt} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \underline{x}$ BECAME $\frac{d\underline{y}}{dt} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \underline{y}$

AFTER MAKING THE CHANGE OF VARIABLES $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$.

Lecture 07
02/01/12

Fundamental subspaces: Kernel and Range (3.5)

$L: V \to W$ linear transformation

Def. • $\text{Ker}(L) = \text{Null}(L) := \{\underline{v} \in V : L(\underline{v}) = \underline{0}\}$.
• $\text{Ran}(L) = \{\underline{w} \in W : L(\underline{v}) = \underline{w}$ for some $v \in V\}$.

Note: • $\text{Ker}(L)$ is solution set of homogeneous eqn.
$L(\underline{v}) = \underline{0}$.
• $\text{Ran}(L)$ is set of $\underline{w} \in W$ s.t. nonhomogeneous eqn.
$L(\underline{v}) = \underline{w}$ has a soln. (i.e, $\underline{w}$ is in span of $\{L(\underline{b}_1), \dots, L(\underline{b}_n)\}$, where $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ is a basis of V.)

Claim: $\text{Ker}(L)$ ~~is~~ a subspace of V and
$\text{Ran}(L)$ is a subspace of W.

Why? (i) $\underline{x}, \underline{y} \in \text{Ker}(L)$, c scalar $\Rightarrow$ $L(\underline{x}+\underline{y}) = \underset{\underline{0}}{L(\underline{x})} + \underset{\underline{0}}{L(\underline{y})} = \underline{0}$
so, $\underline{x}+\underline{y} \in \text{Ker}(L)$
$L(c\underline{x}) = cL(\underline{x}) = \underline{0}$
so, $c\underline{x} \in \text{Ker}(L)$.

(ii) $x, y \in \text{Ran}(L)$, c scalar $\Rightarrow$ there are $\underline{v}_1, \underline{v}_2 \in V$ s.t.
$L(\underline{v}_1) = \underline{x}$, $L(\underline{v}_2) = \underline{y}$.

$\Rightarrow \quad L(\underline{v}_1+\underline{v}_2) = L(\underline{v}_1) + L(\underline{v}_2) = \underline{x}+\underline{y}$

SINCE $\underline{v}_1+\underline{v}_2 \in V$ SINCE V VECTOR SPACE,

$\underline{x}+\underline{y} \in \mathrm{Ran}(L)$.

SIMILARLY, $c\underline{x} \in \mathrm{Ran}(L)$.

$\mathrm{Ker}(L)$ AND $\mathrm{Ran}(L)$ CLOSED UNDER VECTOR ADDITION AND ~~SCALAR~~ MULTIPLICATION $\Rightarrow$ SUBSPACES.

THM. $\mathrm{Ker}(L) = \{0\} \iff L$ IS 1-1
(I.E., $L(\underline{x}) = L(\underline{y}) \Rightarrow \underline{x}=\underline{y}$).

PF. "$\Rightarrow$" SUPPOSE $\mathrm{Ker}(L) = \{0\}$. THEN, IF $L(\underline{x}) = L(\underline{y})$,

$L(\underline{x}-\underline{y}) = \underline{0} \Rightarrow \underline{x}-\underline{y} \in \mathrm{Ker}(L) \Rightarrow \underline{x}-\underline{y} = \underline{0}$

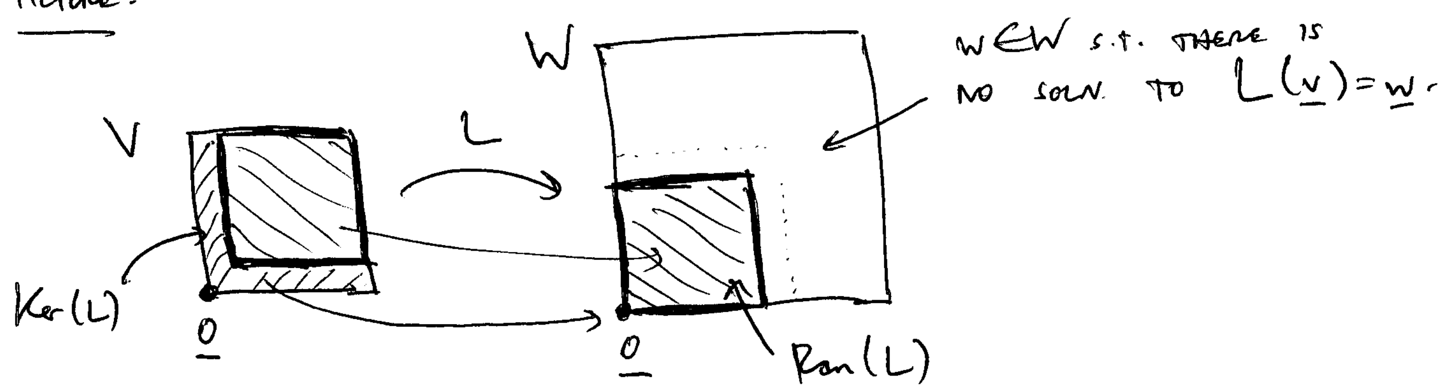
$\Rightarrow \underline{x} = \underline{y}$.

$\Rightarrow L$ IS 1-1.

"$\Leftarrow$" SUPPOSE L IS 1-1. SINCE $L(\underline{0}) = \underline{0}$, IF $L(\underline{v}) = \underline{0}$ FOR SOME $\underline{v} \in V$, THEN $L(\underline{v}) = L(\underline{0})$

$\Rightarrow \underline{v} = \underline{0} \Rightarrow \mathrm{Ker}(L) = \{0\}$.

PICTURE:

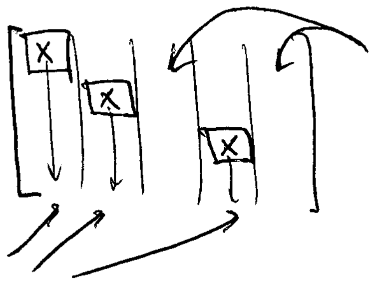


Q: HOW TO FIND BASIS OF $\mathrm{Ker}(L)$, $\mathrm{Ran}(L)$?

A: SUPPOSE $V = \mathbb{R}^n$, $W = \mathbb{R}^m$. THEN, ANY LINEAR TRANSFORMATION $L : V \to W$ TAKES THE FORM $L(\underline{v}) = A\underline{v}$ FOR SOME $A \in M_{m,n}$.

① $\mathrm{Ker}(A) = \{ x \in \mathbb{R}^n : A\underline{x} = \underline{0} \}$.

$$A\underline{x} = \underline{0} \iff A_{rref}\,\underline{x} = \underline{0}$$

FOR EXAMPLE, $A_{rref} =$ 

NO PIVOTS $\Rightarrow$ FREE VARIABLES. x_3 AND x_5.

$$\Rightarrow \underline{x} = x_3 \begin{pmatrix} \\ \end{pmatrix} + x_5 \begin{pmatrix} \\ \end{pmatrix}$$

pivot COLUMNS

FORMS BASIS OF $\mathrm{Ker}(L)$.

LET $\{x_{i_j}\}$ BE FREE VARIABLES — THAT IS, $\{x_{i_1}, \dots, x_{i_k}\}$ CORRESPOND TO COLUMNS <u>WITHOUT</u> PIVOTS.

THEN SOLVING BY BACK SUBSTITUTION GIVES

$$\underline{x} = x_{i_1}\underline{v}_{i_1} + \dots + x_{i_k}\underline{v}_{i_k}.$$

$\Rightarrow \{\underline{v}_{i_1}, \dots, \underline{v}_{i_k}\}$ FORMS A BASIS OF $\mathrm{Ker}(L)$ AFTER DROPPING ANY LINEARLY DEPENDENT VECTORS.

② $\mathrm{Ran}(A) = \{ \underline{b} \in \mathbb{R}^m : \underline{b} \in \mathrm{span}(\text{columns of } A) \}$.

THE COLUMN SPACE OF A HAS A BASIS GIVEN BY THE PIVOT COLUMNS <u>OF THE ORIGINAL MATRIX A</u> (NOT A_{rref}).

EX. $A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 \end{pmatrix}$

SINCE $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 \end{pmatrix} \begin{matrix} \\ -2R_1 \\ -R_1 - R_2 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} +\frac{2}{3}R_2 \\ \times -\frac{1}{3} \\ \end{matrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$

$A_{rref} = \begin{pmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow x_2, x_4$ FREE VARIABLES.

PIVOT COLUMNS OF A_{rref}

(1) $A\underline{x} = \underline{0} \iff A_{rref}\,\underline{x} = \underline{0}$

$\iff \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\iff$ $x_1 = -x_2$, x_2 free, $x_3 = -x_4$, x_4 free $\iff \underline{x} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

SO, $\{(-1,1,0,0)^T, (0,0,-1,1)^T\}$ IS A BASIS OF $Ker(A)$.

(2) $A\underline{x} = \underline{b} \iff \underline{b} \in$ span (columns of A)

SINCE THE PIVOT COLUMNS OF A ARE THE FIRST AND THIRD COLUMNS OF A,

$\{(1,2,3)^T, (2,1,3)^T\}$ IS A BASIS OF $Ran(A)$.

NOTE: IN GENERAL, IF $L : V \to W$

① $L(\underline{v}) = \underline{0} \iff [L(\underline{v})]_{\mathcal{D}} = \underline{0}$

$$\iff \underbrace{[L]_{\mathcal{D}\mathcal{B}}}_{\text{MATRIX}} [\underline{v}]_{\mathcal{B}} = \underline{0}$$

THEREFORE, IF WE FIND A BASIS OF $\mathrm{Ker}([L]_{\mathcal{D}\mathcal{B}})$ THIS GIVES A BASIS FOR $\mathrm{Ker}(L)$ BY MAPPING BACK USING $[\cdot]_{\mathcal{B}}^{-1} : \mathbb{R}^n \to V$.

② $L(\underline{v}) = \underline{w} \iff [L(\underline{v})]_{\mathcal{D}} = [\underline{w}]_{\mathcal{D}}$

$$\iff \underbrace{[L]_{\mathcal{D}\mathcal{B}}}_{\text{MATRIX.}} [\underline{v}]_{\mathcal{B}} = [\underline{w}]_{\mathcal{D}}$$

SO, IF WE FIND A BASIS OF $\mathrm{Ran}([L]_{\mathcal{D}\mathcal{B}})$ THIS GIVES A BASIS FOR $\mathrm{Ran}(L)$ BY MAPPING BACK USING $[\cdot]_{\mathcal{D}}^{-1} : \mathbb{R}^m \to W$.

EX. $V = M_{2,2}$, $W = \mathbb{R}_2[t]$. SUPPOSE $L : V \to W$ IS GIVEN BY

$$L\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1 + 2t + 3t^2$$
$$L\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 1 + 2t + 3t^2$$
$$L\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = 2 + t + 3t^2$$
$$L\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = 2 + t + 3t^2$$

$$\Rightarrow [L]_{\mathcal{D}\mathcal{B}} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

WHERE $\mathcal{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$\mathcal{D} = \{1, t, t^2\}$.

SINCE A BASIS OF $\mathrm{Ker}([L]_{\mathcal{D}\mathcal{B}})$ IS $\{(-1,1,0,0)^T, (0,0,-1,1)^T\}$,

$$\downarrow \quad [\cdot]_{\mathcal{B}}^{-1} \quad \downarrow$$

A BASIS OF $\mathrm{Ker}(L)$ IS $\left\{ \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} \right\}$.

Since a basis of $\mathrm{Ran}([L]_{\mathcal{D}\mathcal{B}})$ is $\{(1,2,3)^T, (2,1,3)^T\}$,

$$\big\downarrow [\cdot]_{\mathcal{D}}^{-1}$$

a basis of $\mathrm{Ran}(L)$ is $\{1+2t+3t^2,\ 2+t+3t^2\}$.

LECTURE 08
02/03/12

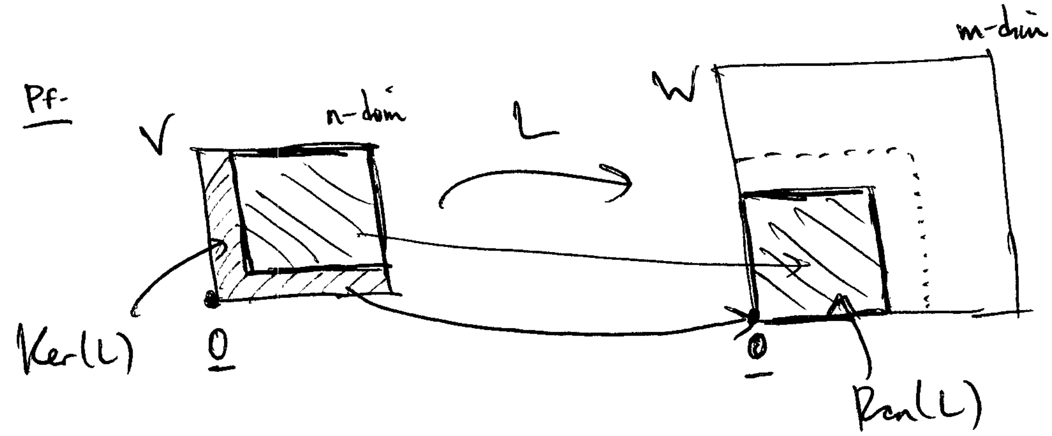
11

$L : V \to W$, $\mathcal{B}$ BASIS OF $V \simeq \mathbb{R}^n$
$\mathcal{D}$ BASIS OF $W \simeq \mathbb{R}^m$.

DEF. $\text{nullity}(L) \doteq \dim(\text{Ker}(L))$.
$\text{rank}(L) \doteq \dim(\text{Ran}(L))$.

THM. (RANK-NULLITY THM.)
$\text{rank}(L) + \text{nullity}(L) = \dim(V)$.

Pf.



$[L]_{\mathcal{D}\mathcal{B}} = m \times n$ MATRIX

$\Rightarrow$ rref OF $[L]_{\mathcal{D}\mathcal{B}}$ HAS k PIVOT COLUMNS AND $n-k$ NON-PIVOT COLUMNS.

$\Rightarrow$ $\text{rank}(L) = k$, $\text{nullity}(L) = n-k$ $\Rightarrow$ $\text{rank} + \text{nullity} = n = \dim(V)$.

#.

CONSEQUENCES:

(RANK-NULLITY THM.)

$$L \text{ is } 1\text{-}1 \iff \text{nullity}(L) = 0 \iff \text{rank}(L) = n.$$

$$L \text{ is ONTO} \iff \text{rank}(L) = m.$$

THEN SINCE

$$\text{rank}(L) = \#\text{ pivots IN } \text{rref}([L]_{\mathcal{D}\mathcal{B}}) \leq \min\{m, n\},$$

THM.

- $n < m \Rightarrow \text{rank}(L) < m \Rightarrow L$ IS NOT ONTO
- $n > m \Rightarrow \text{rank}(L) < n \Rightarrow L$ IS NOT $1\text{-}1$
- $n = m \Rightarrow$ EITHER: (i) $\text{rank}(L) < n$ OR (ii) $\text{rank}(L) = n$ $\Rightarrow$ EITHER: (i) L NOT $1\text{-}1$ NOR ONTO. OR (ii) L BOTH $1\text{-}1$ AND ONTO.

THE LAST STATEMENT GIVES

THM. (FREDHOLM ALTERNATIVE)

$L : V \to V$ LINEAR OPERATOR. THEN, EITHER:

(i) $L(\underline{v}) = \underline{0}$ HAS NONTRIVIAL SOLN'S $\underline{v} \neq \underline{0}$.

OR (ii) $L(\underline{v}) = \underline{b}$ HAS ~~UNIQUE~~ SOLN. FOR ANY $\underline{b} \in V$.

EX. ON $\mathbb{R}_{n-1}[t]$, LET L BE THE OPERATOR

$$L(p(t)) = t\,p'(t) - p(t) \quad \left(\text{I.E. } L = t\frac{d}{dt} - \text{Id.}\right)$$

Q: IS $t\,p'(t) - p(t) = q(t)$ SOLVABLE IN $\mathbb{R}_{n-1}[t]$ FOR ANY $q \in \mathbb{R}_{n-1}[t]$?

A: L IS A LINEAR OPERATOR SUCH THAT $L(t) = t\frac{dt}{dt} - t = \underline{0}$.

THEREFORE, SINCE $t \neq \underline{0}$ SOLVES $L(p(t)) = \underline{0}$ WE HAVE THAT $L(p(t)) = q(t)$ DOES NOT HAVE A SOLN. FOR EVERY q.

REMARK: REMEMBER, WE ARE LOOKING FOR SOLUTIONS $p \in \mathbb{R}_{n-1}[t]$, NOT SIMPLY <u>ANY</u> SOLN'S $p(t)$ TO THE DIFFERENTIAL EQN. $tp'(t) - p(t) = q(t)$! THAT IS, WE CAN'T LOOK FOR SOLUTIONS LIKE e^t, $\sin(kt)$, ETC.