LECTURE 09
02/06/12

EIGENVALUES AND EIGENVECTORS: DEFINITIONS AND NOTATION (4.1-4.2):

$L : V \longrightarrow V$ LINEAR OPERATOR.

DEF. A SCALAR λ IS AN EIGENVALUE OF L IF THERE IS SOME NONZERO $\underline{\xi} \in V$ S.T. $L(\underline{\xi}) = \lambda \underline{\xi}$.
THEN, $\underline{\xi}$ IS CALLED AN EIGENVECTOR CORRESPONDING TO λ.

- SET OF ALL EIGENVALUES OF L IS CALLED THE SPECTRUM OF L, AND IS DENOTED $\sigma(L)$.
- EIGENSPACE CORRESPONDING TO λ IS $E_\lambda \doteq Ker(L - \lambda I)$. ← IDENTITY OPERATOR

MOTIVATION: LET V VECTOR SPACE, WITH BASIS $\mathcal{B}$.

DISCRETE-TIME LINEAR EVOLUTION

$$\begin{cases} \underline{x}(k) = L\,\underline{x}(k-1), & k = 1, 2, \ldots \\ \underline{x}(0) \text{ GIVEN} \end{cases}$$

$$\Downarrow$$

$$\underline{x}(k) = L(L(\cdots L(\underline{x}(0))))$$
$$= (\underbrace{L \circ L \circ \cdots \circ L}_{k \text{ times}})\,\underline{x}(0)$$

$$\Downarrow$$

$$[\underline{x}(k)]_{\mathcal{B}} = [L \circ \cdots \circ L]_{\mathcal{B}} [\underline{x}(0)]_{\mathcal{B}}$$
$$= ([L]_{\mathcal{B}})^k \, [\underline{x}(0)]_{\mathcal{B}}$$

CONTINUOUS-TIME EVOLUTION

$$\left(\begin{cases} \dfrac{d\underline{x}}{dt} = L\,\underline{x}(t) \\ \underline{x}(0) \text{ GIVEN} \end{cases} \right)$$

$$\Downarrow$$

$$\left(\underline{x}(t) = e^{tL}\,\underline{x}(0) \right)$$

WE'LL EXPLAIN WHAT THIS MEANS LATER.

IF $[L]_{\mathcal{B}}$ IS DIAGONAL,

$$[\underline{x}(t)]_{\mathcal{B}} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} [\underline{x}(0)]_{\mathcal{B}}$$

$$[L]_{\mathcal{B}} = \left([L(\underline{b}_1)]_{\mathcal{B}} \cdots [L(\underline{b}_n)]_{\mathcal{B}} \right)$$ SQUARE MATRIX.

NOTE: IF $A \in M_{n,n}$, WHAT IS A^k? IF k LARGE?

SIMPLEST WHEN $A = (d_1 \underline{e}_1 \cdots d_n \underline{e}_n) = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$

SINCE THEN, $A^k = (d_1^k \underline{e}_1 \cdots d_n^k \underline{e}_n) = \begin{pmatrix} d_1^k & & 0 \\ & \ddots & \\ 0 & & d_n^k \end{pmatrix}$.

IDEA: IF $L(\underline{b}_i) = \lambda_i \underline{b}_i$ (I.E., λ_i EIGENVALUE W/ EIGENVECTOR $\underline{b}_i$)

THEN $[L(\underline{b}_i)]_{\mathcal{B}} = [\lambda_i \underline{b}_i]_{\mathcal{B}} = \lambda_i \underline{e}_i$, AND

$$[L]_{\mathcal{B}} = (\lambda_1 \underline{e}_1 \cdots \lambda_n \underline{e}_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

$$\Rightarrow [\underline{x}(k)]_{\mathcal{B}} = \left([L]_{\mathcal{B}}\right)^k [\underline{x}(0)]_{\mathcal{B}} = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} [\underline{x}(0)]_{\mathcal{B}}$$

I.E.,
$$\underbrace{\begin{aligned} x_1(k) &= \lambda_1^k x_1(0) \\ &\vdots \\ x_n(k) &= \lambda_n^k x_n(0) \end{aligned}}_{\text{SIMPLE EXPRESSION FOR COORDINATES OF } \underline{x}(k) \text{ IN } \mathcal{B}!}$$
WHERE $[\underline{x}(k)]_{\mathcal{B}} = (x_1(k), \ldots, x_n(k))^T$.

Q: WHAT IF $L(\underline{d}_i) = \lambda_i \underline{d}_i$ ONLY IN SOME OTHER BASIS $\mathcal{D} = (\underline{d}_1, \ldots, \underline{d}_n)$?

$$[\underline{x}(k)]_{\mathcal{B}} = \left([L]_{\mathcal{B}}\right)^k [\underline{x}(0)]_{\mathcal{B}} = \left(P_{\mathcal{B}\mathcal{D}} [L]_{\mathcal{D}} P_{\mathcal{D}\mathcal{B}}\right)^k [\underline{x}(0)]_{\mathcal{B}}$$

CHANGE OF BASIS.

Ex. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

EIGENVALUES: 3, 1

$(1,1)^T$ $\quad$ $(1,-1)^T$

CHECK: $A\underline{\xi} = \lambda \underline{\xi}$.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ✓$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad ✓$$

Ex. (∞-DIM) $V = \mathbb{R}[t]$, $L(p) = t\dfrac{dp}{dt}$.

EIGENVALUES: $\{k\}_{k=0}^{\infty}$

$\{t^k\}_{k=0}^{\infty}$

CHECK:

$$L(t^k) = t\frac{d}{dt}(t^k) = t\,k\,t^{k-1}$$
$$= k t^k. \quad ✓$$

Ex. (∞-DIM.) $V = C^{\infty}(\mathbb{R})$, $L(f) = \dfrac{d^2 f}{dx^2}$.

EIGENVALUES: $\{-k^2\}_{k=0}^{\infty}$, $\{+k^2\}_{k=0}^{\infty}$

$\{\sin(kx)\}_{k=0}^{\infty}$, $\{\cos(kx)\}_{k=0}^{\infty}$ $\qquad$ $\{e^{-kx}\}_{k=0}^{\infty}$, $\{e^{+kx}\}_{k=0}^{\infty}$.

CHECK:

$$L(\sin(kx)) = \frac{d^2}{dx^2}(\sin(kx)) = -k^2\sin(kx) \quad ✓$$
$$L(\cos(kx)) = \frac{d^2}{dx^2}(\cos(kx)) = -k^2\cos(kx) \quad ✓$$

$$L(e^{-kx}) = \frac{d^2}{dx^2}(e^{-kx}) = +k^2 e^{-kx} \quad ✓$$
$$L(e^{+kx}) = \frac{d^2}{dx^2}(e^{+kx}) = +k^2 e^{+kx} \quad ✓.$$

Characteristic Polynomial (4.3):

Q: How to find eigenvalues of matrix $A \in M_{n \times n}$?

A: $$A \underline{x} = \lambda \underline{x} \iff (A - \lambda I)\underline{x} = 0$$
$$\iff \operatorname{Ker}(A - \lambda I) \neq \{\underline{0}\}$$
$$\iff \det(A - \lambda I) = 0$$

Def. $p_A(\lambda) \doteq \det(A - \lambda I)$ polynomial of degree n. (why?)

- Eigenvalues of A $\iff$ roots of p_A.
  $(A\underline{\xi} = \lambda \underline{\xi})$ $\qquad (p_A(\lambda) = 0)$.

Ex. Find eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

A: $$A - \lambda I = \begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix}$$

① $$p_A(\lambda) = \det(A - \lambda I) = (3-\lambda)\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & 3-\lambda \end{vmatrix} + 4\begin{vmatrix} 2 & -\lambda \\ 4 & 2 \end{vmatrix}$$
$$= (3-\lambda)(\lambda^2 - 3\lambda - 4) + 2(2\lambda + 2) + 4(4\lambda + 4)$$
$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$
$$= (\lambda - 8)(\lambda + 1)^2 = (\lambda - 8)(\lambda + 1)(\lambda + 1).$$

$\Rightarrow$ Eigenvalues are -1 and 8.

② Solve $(A - \lambda I)\underline{\xi} = \underline{0}$ to get eigenvector(s) corresponding to λ.

- For $\lambda = -1$, $$A - (-1)I = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & \frac{1}{2} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $\xi_1 = -\frac{1}{2}\xi_2 - \xi_3$
ξ_2 = free
ξ_3 = free
$\Rightarrow$ $\underline{\xi} = \xi_2 \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + \xi_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

THEREFORE, THE TWO EIGENVECTORS CORRESPONDING TO EIGENVALUE -1 ARE $(-\frac{1}{2}, 1, 0)^T$ AND $(-1, 0, 1)^T$. THE EIGENSPACE E_{-1} CONSISTS OF ALL LINEAR COMBINATIONS OF THESE VECTORS.

- FOR $\lambda = 8$, $A - 8I = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$

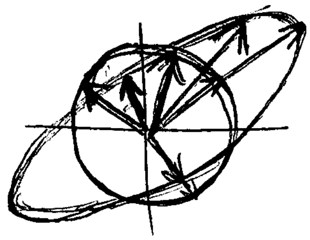
$$\Rightarrow \begin{aligned} \xi_1 &= \xi_3 \\ \xi_2 &= \tfrac{1}{2}\xi_3 \\ \xi_3 &= \text{FREE} \end{aligned} \quad \Rightarrow \quad \underline{\xi} = \xi_3 \begin{pmatrix} 1 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

THEREFORE, EIGENVECTOR ASSOCIATED TO EIGENVALUE 8 IS $(1, \frac{1}{2}, 1)^T$. E_8 CONSISTS OF THE SPAN OF THIS VECTOR.

<u>NOTE</u>: IN THIS EXAMPLE, EIGENVALUE -1 HAD <u>TWO</u> LINEARLY INDEP. EIGENVECTORS.

<u>Q</u>: WHAT IS GEOMETRIC MEANING OF EIGENVALUES / EIGENVECTORS?

<u>A</u>: FOR EX, CONSIDER $A \in M_{2,2}$ S.T. EIGENVALUES ARE IN $\mathbb{R}$. THEN



- EIGENVECTORS CORRESPOND TO <u>DIRECTIONS</u> OF STRETCHING / COMPRESSION,
- EIGENVALUES MEASURE DEGREE OF STRETCHING $(+)$ OR COMPRESSION $(-)$.

- LATER, WE WILL SEE WHAT IMAGINARY EIGENVALUES MEAN.

LECTURE 10
02/08/12

Q: HOW TO FIND EIGENVALUES / EIGENVECTORS OF $L: V \to V$?

A: USE MATRIX REPRESENTATION. IF V HAS BASIS $\mathcal{D}$,

$$\begin{array}{c} \lambda \text{ EIGENVALUE OF } L \\ \text{W/ CORRESPONDING EIGENVECTOR } \underline{v} \end{array} \iff \begin{array}{c} \lambda \text{ EIGENVALUE OF } [L]_{\mathcal{D}} \\ \text{W/ CORRESPONDING EIGENVECTOR } [\underline{v}]_{\mathcal{D}}. \end{array}$$

PF: $L\underline{v} = \lambda \underline{v} \iff [L\underline{v}]_{\mathcal{D}} = [\lambda \underline{v}]_{\mathcal{D}} \iff [L]_{\mathcal{D}} [\underline{v}]_{\mathcal{D}} = \lambda [\underline{v}]_{\mathcal{D}}.$

- THEREFORE, SPECTRUM $\sigma(L)$ OF L IS INDEPENDENT OF CHOICE OF BASIS ON V. THAT IS, IF $\mathcal{B}, \mathcal{D}$ BASES ON V,

$$[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{D}} [L]_{\mathcal{D}} P_{\mathcal{B}\mathcal{D}}^{-1}$$

HAVE SAME EIGENVALUES
(BUT DIFFERENT EIGENVECTORS).

THIS MOTIVATES THE FOLLOWING RESULT:

DEF. WE SAY A IS CONJUGATE (OR SIMILAR) TO B IF $A = PBP^{-1}$ FOR SOME P.

THM. IF A IS CONJUGATE TO B (I.E., $A = PBP^{-1}$),

$$\begin{array}{c} \lambda \text{ EIGENVALUE OF } B \\ \text{W/ EIGENVECTOR } \underline{v} \end{array} \iff \begin{array}{c} \lambda \text{ EIGENVALUE OF } A \\ \text{W/ EIGENVECTOR } P\underline{v}. \end{array}$$

PF. "$\Rightarrow$" $B\underline{v} = \lambda \underline{v} \Rightarrow A(P\underline{v}) = PB\cancel{P^{-1}P}\underline{v} = P\lambda\underline{v} = \lambda(P\underline{v})$. ✓

"$\Leftarrow$" $A(P\underline{v}) = \lambda(P\underline{v}) \Rightarrow B\underline{v} = P^{-1}AP\underline{v} = P^{-1}\lambda P\underline{v} = \lambda \underline{v}$. ✓

WE CAN USE THIS TO PROVE ONE OF THE MAIN RESULTS OF THE COURSE:

<u>THM.</u> (DIAGONALIZATION)

$A \in M_{n,n}$ HAS n LINEARLY INDEP. EIGENVECTORS $\mathcal{B} = \{\underline{\xi}_1, \dots, \underline{\xi}_n\}$ CORRESPONDING TO EIGENVALUES $\lambda_1, \dots, \lambda_n$

$$\iff \quad A = PDP^{-1} \quad \text{WITH} \quad D = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$P = \begin{pmatrix} \underline{\xi}_1 & \cdots & \underline{\xi}_n \end{pmatrix}$$

I.E, ith COLUMN IS ith EIGENVECTOR

- <u>MAIN CONSEQUENCE:</u> $L : V \to V$ LINEAR OPERATOR IS <u>DIAGONALIZABLE</u> (I.E, CAN BE REPRESENTED AS A DIAGONAL MATRIX FOR SOME BASIS) $\iff$ EIGENVECTORS OF L FORM A BASIS OF V.

<u>PF. OF THM.</u>

"$\Rightarrow$" LET $\mathcal{E} = \{\underline{e}_1, \dots, \underline{e}_n\}$ BE THE STD. BASIS IN $\mathbb{R}^n$.

$$A = [A]_{\mathcal{E}} = P_{\mathcal{E}\mathcal{B}} \, [A]_{\mathcal{B}} \, P_{\mathcal{E}\mathcal{B}}^{-1} \quad \text{WHERE}$$

$$[A]_{\mathcal{B}} = \Big(\underbrace{[A\underline{\xi}_1]_{\mathcal{B}}}_{\lambda_1 \underline{\xi}_1} \cdots \underbrace{[A\underline{\xi}_n]_{\mathcal{B}}}_{\lambda_n \underline{\xi}_n} \Big) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = D$$

$$P_{\mathcal{E}\mathcal{B}} = \left([\underline{\xi}_1]_{\mathcal{E}} \cdots [\underline{\xi}_n]_{\mathcal{E}} \right) = \left(\underline{\xi}_1 \cdots \underline{\xi}_n \right) = P.$$

"$\Leftarrow$" $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ HAS EIGENVALUES $\lambda_1, \dots, \lambda_n$

w/ CORRESPONDING EIGENVECTORS $\{\underline{e}_i\}_{i=1}^n$. THEN BY PREVIOUS THM, $A = PDP^{-1}$ HAS SAME EIGENVALUES, BUT WITH CORRESPONDING EIGENVECTORS $\{P\underline{e}_i\}_{i=1}^n$.

$P\underline{e}_i$ IS SIMPLY THE i^{th} COLUMN OF P. SINCE P IS INVERTIBLE BY HYPOTHESIS, $\{P\underline{e}_i\}_{i=1}^n$ ARE LINEARLY INDEPENDENT SINCE COLUMNS OF AN INVERTIBLE MATRIX MUST BE LINEARLY INDEP.

__EX.__ $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1$$

$$= \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1)$$

$p_A(\lambda) = 0 \Rightarrow$ EIGENVALUES $\lambda = 3$, $\lambda = 1$.

__FOR $\lambda = 3$:__ $A - \lambda I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ BASIS OF $\operatorname{Ker}(A - 3I)$ IS $\underline{x} = x_2(1,1)^T$.

__FOR $\lambda = 1$:__ $A - \lambda I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ BASIS OF $\operatorname{Ker}(A - I)$ IS $\underline{x} = x_2(1,-1)^T$.

SINCE $(1,1)^T$ AND $(1,-1)^T$ ARE LINEARLY INDEP., WE MUST HAVE THAT

$$\underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}}_{P^{-1}}.$$

LECTURE 11
02/10/12

LAST TIME WE SAW THAT $A \in M_{n,n}$ IS DIAGONALIZABLE (I.E, $A = PDP^{-1}$ FOR SOME DIAGONAL MATRIX D AND INVERTIBLE MATRIX P) $\iff$ A HAS n LINEARLY INDEP. EIGENVECTORS.

- DOES THIS MEAN THAT ALL MATRICES ARE DIAGONALIZABLE, AND WHAT ISSUES MUST WE DEAL WITH IF WE WANT TO DIAGONALIZE A MATRIX?

A FEW EXAMPLES:

EX. $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 2^2$$

$\Rightarrow p_A(\lambda)$ HAS NO REAL ROOTS! $p_A(\lambda) = 0 \Rightarrow$

$\lambda = 1 + 2i, \ \lambda = 1 - 2i$ WITH $i = \sqrt{-1} \in \mathbb{C}$.

SO, NEED TO CONSIDER COMPLEX EIGENVALUES AND EIGENVECTORS TO HAVE ANY HOPE OF FINDING ENOUGH LINEARLY INDEP. EIGENVECTORS OF A $\Rightarrow$ $\boxed{\text{COMPLEXIFICATION}}$.

EX. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$p_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 \Rightarrow \lambda = 0$$

ONLY EIGENVALUE.

$A - 0I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow$ BASIS OF $\mathrm{Ker}(A - 0I)$ IS $(1, 0)^T$.

SO WE ONLY HAVE <u>ONE</u> EIGENVECTOR FOR A AND

$$A \neq P D P^{-1} \quad \text{FOR } D \text{ DIAGONAL!}$$

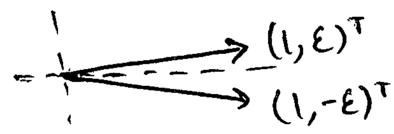
HOWEVER, NOTE THAT THIS PATHOLOGICAL CASE CAN BE RESOLVED IF WE SLIGHTLY PERTURB A:

$$A_\varepsilon \doteq \begin{pmatrix} 0 & 1 \\ \varepsilon^2 & 0 \end{pmatrix} \quad \text{WITH } \varepsilon > 0 \text{ SMALL AS DESIRED.}$$

$$p_{A_\varepsilon}(\lambda) = \det(A_\varepsilon - \lambda I) = \lambda^2 - \varepsilon^2 = 0 \Rightarrow \lambda = \varepsilon, \ \lambda = -\varepsilon$$

(I.E, THE ZERO EIGENVALUE SPLIT INTO TWO DISTINCT EIGENVALUES)

$$\begin{cases} \ker(A_\varepsilon - \varepsilon I) \text{ HAS BASIS } (1, -\varepsilon)^T \\ \ker(A_\varepsilon + \varepsilon I) \text{ HAS BASIS } (1, \varepsilon)^T \end{cases}$$



(I.E, THE EIGENVECTOR $(1,0)^T$ SPLIT INTO TWO DISTINCT EIGENVECTORS)

THEREFORE,

$$A_\varepsilon = P_\varepsilon D_\varepsilon P_\varepsilon^{-1} \quad \text{WITH} \quad D_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

$$P_\varepsilon = \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon \end{pmatrix}$$

AND THE ORIGINAL MATRIX A IS <u>ALMOST</u> DIAGONALIZABLE.

TO RECAP, IN THIS EXAMPLE THE EIGENVALUE $\lambda = 0$ WAS A DOUBLE ROOT OF $p_A(\lambda)$, BUT ITS ASSOCIATED EIGENSPACE E_0 HAD ONLY ONE DIMENSION. THIS WILL LEAD US TO A DISCUSSION ON MULTIPLICITY OF EIGENVALUES AND $\boxed{\text{JORDAN CANONICAL FORM}}$.