LECTURE 15
02/22/12

TRICKS FOR FINDING / VERIFYING EIGENVALUES (4.6):

- IF $A \in M_{n,n}$ TRIANGULAR (UPPER- OR LOWER-TRIANGULAR), EIGENVALUES OF A ARE DIAGONAL ENTRIES $\{A_{ii}\}_{i=1}^{n}$.
- IF $A \in M_{n,n}(\mathbb{R})$ (I.E., A REAL MATRIX), $\lambda = a+ib$ EIGENVALUE $\Longleftrightarrow$ $\bar{\lambda} = a-ib$ EIGENVALUE.
- DEFINE TRACE $Tr(A) \doteq \sum_{i=1}^{n} A_{ii}$. (SUM OF DIAGONAL ENTRIES). THEN, $Tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ (SUM OF EIGENVALUES, COUNTING MULTIPLICITIES).

  Pf. EVEN THOUGH $AB \neq BA$ GENERALLY (I.E., MATRICES TYPICALLY DON'T COMMUTE), WE STILL HAVE THAT $Tr(AB) = Tr(BA)$. THEN, SINCE A HAS JORDAN FORM $A = P\tilde{D}P^{-1}$,

$$Tr(A) = Tr(\underbrace{P\tilde{D}}_{\text{CALL } B_1}\underbrace{P^{-1}}_{\text{CALL } B_2}) = Tr(\underbrace{P^{-1}}_{B_2}\underbrace{P\tilde{D}}_{B_1}) = Tr(\tilde{D})$$
$$= \lambda_1 + \cdots + \lambda_n.$$

- $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$ (DETERMINANT IS PRODUCT OF EIGENVALUES)

  Pf. SINCE $\det(AB) = \det(A)\det(B)$ FOR ANY $A, B \in M_{n,n}$,

$$\det(A) = \det(P\tilde{D}P^{-1}) = \cancel{\det(P)}\det(\tilde{D})\cancel{\det(P^{-1})} = \det(\tilde{D})$$
$$\det(P^{-1}) = \frac{1}{\det(P)}$$
$$= \lambda_1 \cdots \lambda_n.$$

EX. WHAT ARE EIGENVALUES OF $A = \begin{pmatrix} -6 & 7 \\ 4 & 6 \end{pmatrix}$?

$$\left.\begin{aligned} Tr(A) &= 0 = \lambda_1 + \lambda_2 \\ \det(A) &= -36 - 28 = -64 = \lambda_1 \lambda_2 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_1 &= 8 \\ \lambda_2 &= -8 \end{aligned}$$

NOTE: FOR MOST PROBLEMS, USE THESE TRICKS TO VERIFY YOUR ANSWERS ARE CORRECT.

EVOLUTION PROBLEMS (5.1, 5.2):

① DISCRETE-TIME EVOLUTION:

$$\begin{cases} \underline{x}(k) = A\,\underline{x}(k-1) \\ \underline{x}(0) \text{ known} \end{cases}, \qquad A \text{ GIVEN.}$$

STATE OF SYSTEM AT TIME $k \in \mathbb{N}$.

$$\underbrace{\underline{x}(k)}_{\text{STATE}} \in \mathbb{R}^n \text{ FOR ALL } k = 0, 1, 2, \ldots$$

$$\underline{x}(k) = \underbrace{(x_1(k), x_2(k), \ldots, x_n(k))^T}_{\text{COMPONENTS OF STATE VECTOR } \underline{x}(k)}$$

$\boxed{\text{SOL'N:}}$ $\underline{x}(k) = A\,\underline{x}(k-1) = A(A\,x(k-2)) = \ldots = \underbrace{A^k}_{\text{NEEDS TO BE DETERMINED}} \underline{x}(0).$

② CONTINUOUS-TIME EVOLUTION:

$$\begin{cases} \dfrac{d\underline{x}(t)}{dt} = A\,\underline{x}(t) \\ \underline{x}(0) \text{ known} \end{cases}, \qquad A \text{ GIVEN.}$$

STATE OF SYSTEM AT TIME $t \in [0, \infty)$.

$$\underbrace{\underline{x}(t)}_{\text{STATE}} \in \mathbb{R}^n \text{ FOR ALL } t \geq 0.$$

$$\underline{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T.$$

THIS IS KNOWN AS ~~A~~ SYSTEM OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODE).

$\boxed{\text{SOL'N:}}$ IF $n=1$, I.E, IF A IS A ~~SCALAR~~ SCALAR, SAY a, THEN THE EVOLUTION PROBLEM IS EASILY SOLVED.

$$\begin{cases} \frac{dx(t)}{dt} = a x(t) \\ x(0) \text{ known} \end{cases} \Rightarrow \int \frac{dx(t)}{x(t)} = \int a\,dt \Rightarrow \ln(x(t)) - \ln(x(0)) = ta$$

$$\Rightarrow x(t) = e^{ta} x(0).$$

(TO CHECK THIS IS CORRECT, SUBSTITUTE INTO THE ODE TO FIND THAT $\frac{d}{dt}\big(\underbrace{e^{ta}x(0)}_{x(t)}\big) = a e^{ta} x(0) = a x(t)$. ✓)

BY ANALOGY, IF $n > 1$ THEN $A \in M_{n,n}$ IS A MATRIX AND THE SOLN. IS

$$\underline{x}(t) = \underbrace{e^{tA}}\,\underline{x}(0).$$

WE WILL DEFINE THIS LATER USING MATRIX EXPONENTIALS, WHICH WILL DETERMINE THIS TO BE

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{A^k}.$$

NEEDS TO BE DETERMINED.

<u>NOTE:</u> FOR BOTH DISCRETE- AND CONTINUOUS-TIME EVOLUTIONS WE NEED TO FIND A^k, FOR SOME GIVEN $k \in \mathbb{N}$. FOR LARGE k, THIS CAN BE DIFFICULT — UNLESS WE DIAGONALIZE A! IF A IS DIAGONALIZABLE, THEN

$$A^k = (PDP^{-1})^k = (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cdots(\cancel{P}DP^{-1})$$

$$= PD^kP^{-1}, \text{ WHERE } D^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}.$$

LECTURE 16
02/24/12

DISCRETE-TIME EVOLUTION:

EX (POPULATION GROWTH, FIBONACCI SEQUENCE)

LET $x_1(t)$, $x_2(t)$ BE THE NUMBER OF JUVENILE AND ADULT RABBITS THAT LIVE IN A CERTAIN REGION. THESE POPULATIONS EVOLVE ACCORDING TO THE FOLLOWING RULES:

- EVERY MONTH, EACH ADULT GIVES BIRTH TO ONE JUVENILE.
- JUVENILES GROW INTO ADULTS IN ONE MONTH
- ADULTS / JUVENILES DO NOT DIE.

THEN, FOR EACH MONTH $k = 0, 1, 2, \dots$,

$$\Delta x_1(k) \doteq x_1(k) - x_1(k-1) = \overbrace{x_2(k-1)}^{\text{BIRTH OF NEW JUVENILES}} - \overbrace{x_1(k-1)}^{\text{JUVENILES BECOME ADULTS}}$$

$$\Delta x_2(k) \doteq x_2(k) - x_2(k-1) = \overbrace{x_1(k-1)}^{\text{JUVENILES BECOME ADULTS}}.$$

LETTING $\underline{x}(k) = (x_1(k), x_2(k))^T$, WE HAVE THAT.

$$\underline{x}(k) = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}_{A} \underline{x}(k-1).$$

$$\Rightarrow p_A(\lambda) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\Rightarrow \lambda_1 = \underbrace{\frac{1+\sqrt{5}}{2}}_{\approx 1.618 \text{ "GOLDEN RATIO"}}, \quad \lambda_2 = \underbrace{\frac{1-\sqrt{5}}{2}}_{\approx -0.618}.$$

THE CORRESPONDING EIGENSPACES ARE

$E_{\frac{1+\sqrt{5}}{2}}$:

$$\begin{pmatrix} -\left(\frac{1+\sqrt{5}}{2}\right) & 1 \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \times \frac{1-\sqrt{5}}{2} \rightarrow \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} -R_1$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1-\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \left(-(1-\sqrt{5}), 2\right)^T \right\} \text{ BASIS.}$$

$E_{\frac{1-\sqrt{5}}{2}}$:

$$\begin{pmatrix} -\left(\frac{1-\sqrt{5}}{2}\right) & 1 \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \times \frac{1+\sqrt{5}}{2} \rightarrow \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} -R_1$$

$$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow \left\{ \left(-(1+\sqrt{5}), 2\right)^T \right\} \text{ BASIS.}$$

LET $\begin{cases} \underline{b}_1 = \left(-(1-\sqrt{5}), 2\right)^T \\ \underline{b}_2 = \left(-(1+\sqrt{5}), 2\right)^T \end{cases}$ BE THE BASIS $\mathcal{B}$ OF EIGENVECTORS OF A.

THEN,

$$\begin{tikzcd}
(a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}} \arrow[r, "D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}"] & {[\underline{x}(k)]_{\mathcal{B}}} = (a_1(k), a_2(k))^T \arrow[d, "P = (\underline{b}_1 \ \underline{b}_2)"] \\
(x_1(0), x_2(0))^T = \underline{x}(0) \arrow[u, "P^{-1}"] \arrow[r, dashed, "A^k"'] & \underline{x}(k) = (x_1(k), x_2(k))^T
\end{tikzcd}$$

THAT IS, $\underline{x}(k) = a_1(k)\, \underline{b}_1 + a_2(k)\, \underline{b}_2$, WITH $\begin{pmatrix} a_1(0) \\ a_2(0) \end{pmatrix} = [\underline{x}(0)]_{\mathcal{B}}$

$$a_1(k) = \lambda_1^k a_1(0), \qquad a_2(k) = \lambda_2^k a_2(0)$$

FOR EXAMPLE, SUPPOSE WE START WITH ONE JUVENILE AND NO ADULTS. THEN,

$$\underline{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow [\underline{x}(0)]_{\mathcal{B}} = \Big(\underset{a_1(0)}{\underset{\shortparallel}{\frac{1}{2\sqrt{5}}}} , \; \underset{a_2(0)}{\underset{\shortparallel}{-\frac{1}{2\sqrt{5}}}} \Big)^T .$$

$$\Rightarrow \underline{x}(k) = \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^k}_{\substack{\to \infty \text{ AS} \\ k \to \infty .}} \frac{1}{2\sqrt{5}} \underline{b}_1 + \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^k}_{\substack{\to 0 \text{ AS} \\ k \to \infty}} \left(-\frac{1}{2\sqrt{5}}\right) \underline{b}_2$$

$\Rightarrow \underline{b}_1$ "UNSTABLE MODE" $\qquad \Rightarrow \underline{b}_2$ "STABLE MODE".

FOR k LARGE

$$\approx \left(\frac{1+\sqrt{5}}{2}\right)^k \frac{1}{2\sqrt{5}} \underline{b}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{k} \end{pmatrix} .$$

<u>CONSEQUENCES (FOR LARGE k):</u>

- POP. OF ADULTS WILL BE APPROX. $\frac{1+\sqrt{5}}{2} \approx 1.618$ TIMES MORE THAN THAT OF JUVENILES.
- BOTH POPULATIONS GROW EXPONENTIALLY, APPROX. BY 1.618 TIMES EACH MONTH.
- TOTAL POPULATION OF RABBITS IS APPROX.

$$p(k) = x_1(k) + x_2(k) = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} .$$

CONTINUOUS-TIME EVOLUTION:

LAST TIME, WE SAW THAT THE SOLN. TO

$$\begin{cases} \frac{d\underline{x}}{dt} = A\underline{x} \\ \underline{x}(0) \text{ given} \end{cases} \quad \text{is} \quad \underline{x}(t) = e^{tA}\,\underline{x}(0).$$

WHAT IS THIS?

MATRIX EXPONENTIAL (4.8):

FOR SCALAR $a \in \mathbb{R}$, THE POWER SERIES OF e^a IS

$$e^a = 1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{a^k}{k!}.$$

FOR $A \in M_{n,n}$, DEFINE THE MATRIX EXPONENTIAL

$$e^A \doteq I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

(FOR $L: V \to V$, CAN DEFINE $L^n \doteq L \circ L \circ \cdots \circ L$ (n TIMES)
AND $e^L = I + \frac{L}{1!} + \frac{L^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{L^k}{k!}$.)

- IF A IS DIAGONALIZABLE, $A = PDP^{-1}$ AND

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=0}^{\infty} \frac{t^k}{k!}(PD^kP^{-1})$$

$$= P\left(\sum_{k=0}^{\infty} \frac{t^k}{k!} D^k\right)P^{-1} = P e^{tD} P^{-1}.$$

NOTE THAT FOR D DIAGONAL,

$$e^{tD} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}.$$

THEREFORE, IF A IS DIAGONALIZABLE,

$$\underline{x}(t) = e^{tA}\,\underline{x}(0)$$
$$= P\,e^{tD}\,P^{-1}\,\underline{x}(0)$$

$$\begin{tikzcd}[column sep=huge, row sep=large]
{[\underline{x}(0)]_{\mathcal{B}}} \arrow[r, "e^{tD} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}"'] & {[\underline{x}(t)]_{\mathcal{B}}} \arrow[d, "P = (\underline{b}_1 \cdots \underline{b}_n)"] \\
\underline{x}(0) \arrow[u, "P^{-1}"] \arrow[r, dashed, "e^{tA}"'] & \underline{x}(t)
\end{tikzcd}$$

WHERE $\{\underline{b}_1, \dots, \underline{b}_n\} = \mathcal{B}$ IS THE BASIS OF EIGENVECTORS.

Lecture 17
02/27/12

Recall:

① Discrete-time evolution

$$\begin{cases} \underline{x}(k) = A\,\underline{x}(k-1) \\ \underline{x}(0) \text{ known} \end{cases} \Longrightarrow \quad \begin{aligned} \underline{x}(k) &= A^k\,\underline{x}(0) \\ &= P D^k P^{-1}\,\underline{x}(0) \end{aligned}$$

if $A = PDP^{-1}$.

Ex. $(n=2)$

$$\begin{tikzcd} (a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}} \arrow[rr, "D^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}"] & & {[\underline{x}(k)]_{\mathcal{B}}} \arrow[d, "P"] \\ (x_1(0), x_2(0))^T = \underline{x}(0) \arrow[u, "P^{-1}"] \arrow[rr, "A^k"'] & & \underline{x}(k) \end{tikzcd}$$

$$[\underline{x}(k)]_{\mathcal{B}} = (a_1(k), a_2(k))^T = (\lambda_1^k a_1(0), \lambda_2^k a_2(0))^T.$$

with $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ basis of e-vectors of A.

② Cont. time evolution

$$\begin{cases} \frac{dx(t)}{dt} = A\,x(t) \\ x(0) \text{ known} \end{cases} \Longrightarrow \quad \begin{aligned} \underline{x}(t) &= e^{tA}\,\underline{x}(0) \\ &= P e^{tD} P^{-1}\,\underline{x}(0) \end{aligned}$$

if $A = PDP^{-1}$.

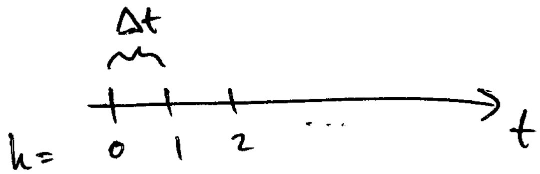
Ex. $(n=2)$

$$\begin{tikzcd} (a_1(0), a_2(0))^T = [\underline{x}(0)]_{\mathcal{B}} \arrow[rr, "e^{tD} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}"] & & {[\underline{x}(t)]_{\mathcal{B}}} \arrow[d, "P"] \\ (x_1(0), x_2(0))^T = \underline{x}(0) \arrow[u, "P^{-1}"] \arrow[rr, "e^{tA}"'] & & \underline{x}(t) \end{tikzcd}$$

$$[\underline{x}(t)]_{\mathcal{B}} = (a_1(t), a_2(t))^T = (e^{t\lambda_1} a_1(0), e^{t\lambda_2} a_2(0))^T$$

with $\mathcal{B} = \{\underline{b}_1, \underline{b}_2\}$ basis of e-vectors of A

NOTE: (2) IS A LIMIT OF (1).



RESTRICT TIMES TO $t = k\Delta t,\ k \in \mathbb{N}$

$$\Rightarrow \frac{d\underline{x}(t)}{dt} \approx \frac{\underline{x}((k+1)\Delta t) - \underline{x}(k\Delta t)}{\Delta t}$$

$$\parallel$$

$$A\underline{x}(t) \approx A\,\underline{x}(k\Delta t).$$

$$\Rightarrow \begin{cases} \underline{x}((k+1)\Delta t) = (I + (\Delta t)A)\,\underline{x}(k\Delta t) \\ \underline{x}(0) \text{ KNOWN} \end{cases}$$

$$\Rightarrow \underline{x}(k\Delta t) = (I + (\Delta t)A)^k\,\underline{x}(0).$$

NOTE NOW THAT AS $\Delta t \to 0$,

$$(I + (\Delta t)A)^k = (I + (\Delta t)A)^{\frac{k\Delta t}{\Delta t}}$$

$$= (I + (\Delta t)A)^{\frac{t}{\Delta t}}$$

$$\xrightarrow[\Delta t \to 0]{} e^{tA}$$

(THIS IS THE ANALOGUE OF THE DEFINITION OF THE EXPONENTIAL:

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x. \quad)$$

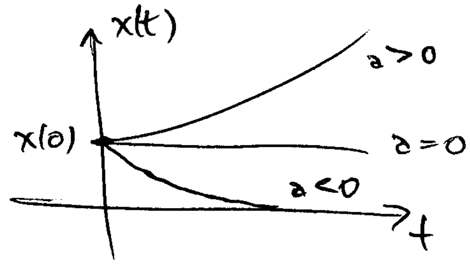
THAT IS, WE CAN THINK OF CONTINUOUS-TIME SYSTEMS (SYSTEMS OF LINEAR ODE) AS LIMITS OF DISCRETE-TIME SYSTEMS.

Ex. (SCALAR EXPONENTIAL GROWTH / DECAY WITH RATE a)

$$\begin{cases} \dfrac{dx}{dt} = ax \\ x(0) \text{ GIVEN} \end{cases} \Rightarrow \quad x(t) = e^{ta}\, x(0)$$

$$\xrightarrow{t \to \infty} \begin{cases} \pm\infty, & a > 0 \\ 0, & a < 0 \\ x(0), & a = 0 \end{cases}$$

IF $x(0) \neq 0$.



Ex. (RADIOACTIVE DECAY W/ MULTIPLE POPULATIONS)

$$X \xrightarrow{r_1} Y, \quad Y \xrightarrow{r_2} Z \qquad \text{WITH } r_1, r_2 \geq 0,\ r_1 \neq r_2.$$

$$\Rightarrow \begin{cases} \dot{x}_1 = -r_1 x_1 \\ \dot{x}_2 = r_1 x_1 - r_2 x_2 \end{cases}$$

WHERE $x_1(t), x_2(t)$ ARE POPULATIONS OF X, Y AT TIME $t \geq 0$, AND $\bullet = \frac{d}{dt}$.

$$\Rightarrow \frac{d\underline{x}(t)}{dt} = A\,\underline{x}(t) \qquad \text{WITH } A = \begin{pmatrix} -r_1 & 0 \\ r_1 & -r_2 \end{pmatrix}$$

$$\underline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

E-VALUES / E-VECTORS ARE

$$\lambda_1 = -r_1, \quad E_{-r_1} = \{\overbrace{((r_2 - r_1), r_1)^T}^{\underline{b}_1}\}$$

$$\lambda_2 = -r_2, \quad E_{-r_2} = \{\underbrace{(0, 1)^T}_{\underline{b}_2}\}.$$

SUPPOSE $\underline{x}(0) = (1, 0)^T$. THEN,

$$\underline{x}(0) = \frac{1}{r_2 - r_1}\,\underline{b}_1 - \frac{r_1}{r_2 - r_1}\,\underline{b}_2$$

AND

$$\underline{x}(t) = \frac{e^{-r_1 t}}{r_2 - r_1}\,\underline{b}_1 - \frac{r_1 e^{-r_2 t}}{r_2 - r_1}\,\underline{b}_2$$

$$= \Big(\underbrace{e^{-r_1 t}}_{x_1(t)},\ \underbrace{\frac{r_1}{r_2 - r_1}\left(e^{-r_1 t} - e^{-r_2 t}\right)}_{x_2(t)}\Big)^T.$$

NOTE THAT AS $r_1 \to r_2$ WE CAN STILL RECOVER A SOLUTION FOR $\underline{x}(t)$:

$$\lim_{r_1, r_2 \to r} \underline{x}(t) = \left(e^{-rt},\ rt\,e^{-rt}\right)^T.$$

IN THIS CASE, A IS NOT DIAGONALIZABLE WHEN $r_1 = r_2 = r$, BUT IT IS ALMOST DIAGONALIZABLE.