LECTURE 25
03/26/12

DUALITY (6.3):

- IF $V = \mathbb{C}^n$ W/ STD. INNER PROD. $\langle \underline{x} | \underline{y} \rangle = \underline{x}^* \underline{y}$, THEN $|\underline{y}\rangle = \underline{y}$ AND $\langle \underline{x} | = \underline{x}^*$.

Q: WHAT IF $V = \mathbb{C}^n$ W/ NONSTANDARD INNER PROD.?

EX. $V = \mathbb{C}^2$, $\langle \underline{x} | \underline{y} \rangle = 2\bar{x}_1 y_1 + 3\bar{x}_2 y_2$.

$$\Rightarrow |\underline{y}\rangle = (y_1, y_2)^T \in \mathbb{C}^2.$$
$$\langle \underline{x} | = (2\bar{x}_1, 3\bar{x}_2)^T \in (\mathbb{C}^2)'.$$

- IF V GENERAL INNER PROD. SPACE W/ BASIS $\mathcal{B} = \{b_i\}_{i=1}^n$,

$$\langle \underline{x} | \underline{y} \rangle_{\mathcal{B}} = \langle \sum_{i=1}^n a_i \underline{b}_i \,|\, \sum_{j=1}^n c_j \underline{b}_j \rangle$$

$$= \sum_{i=1}^n \bar{a}_i \sum_{j=1}^n c_j \underbrace{\langle \underline{b}_i | \underline{b}_j \rangle}_{\text{n×n HERMITIAN MATRIX, CALL } G_{\mathcal{B}}}$$

$$= \overline{(a_1, \dots, a_n)}\, G_{\mathcal{B}}\, (c_1, \dots, c_n)^T$$

$$= \overline{[\underline{x}]_{\mathcal{B}}}^T G_{\mathcal{B}} [\underline{y}]_{\mathcal{B}}$$

WE CALL $G_{\mathcal{B}}$ THE <u>METRIC</u>. BY DEFINITION,

$$(G_{\mathcal{B}})_{ij} = \langle \underline{b}_i | \underline{b}_j \rangle$$

SO $G_{\mathcal{B}}^* = G_{\mathcal{B}}$. BY THE EXPRESSION ABOVE, WE HAVE THAT

$$|y\rangle_{\mathcal{B}} = [\underline{y}]_{\mathcal{B}} \in \mathbb{C}^n$$
$$_{\mathcal{B}}\langle x| = \overline{[\underline{x}]_{\mathcal{B}}}^T G_{\mathcal{B}} \in (\mathbb{C}^n)'.$$

<u>NOTE</u>: IF $\mathcal{B} = \{b_i\}_{i=1}^n$ IS AN ORTHONORMAL BASIS THEN

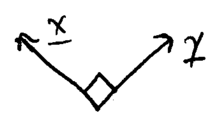
$$G_{\mathcal{B}} = I \quad \text{AND} \quad {}_{\mathcal{B}}\langle x| = \overline{[\underline{x}]_{\mathcal{B}}}^T = [\underline{x}]_{\mathcal{B}}^*.$$

IN THIS CASE, THE INNER PRODUCT IS <u>STANDARD</u>.

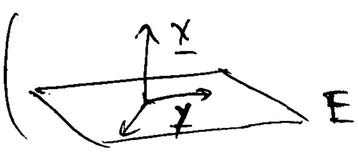
<u>Q</u>: WHAT DO WE MEAN BY AN ORTHONORMAL BASIS?

<u>ORTHOGONALITY AND ORTHONORMAL BASES (6.4):</u>

<u>DEF.</u> FOR $\underline{x}, \underline{y} \in V$, $\underline{x}$ IS <u>ORTHOGONAL</u> TO $\underline{y}$ (DENOTE $\underline{x} \perp \underline{y}$) IF $\langle \underline{x} | \underline{y} \rangle = 0$.

( $V = \mathbb{R}^n$.)

- IF $E \subset V$ SUBSPACE, $\underline{x} \perp E$ IF $\underline{x} \perp \underline{y}$ $\forall \underline{y} \in E$.

()

Ex. FIND ALL VECTORS ORTHOGONAL TO $\underline{x} = (i, 2, 1+i)^T$.

$$\langle \underline{x} | \underline{y} \rangle = \underline{x}^* \underline{y} = 0 \implies (-i, 2, 1-i)(y_1, y_2, y_3)^T = 0.$$

Ex. FIND ALL VECTORS ORTHOGONAL TO BOTH $\underline{x}_1 = (i, 2, 1+i)^T$ AND $\underline{x}_2 = (3, 2-i, i)^T$.

$$\left.\begin{aligned} \langle \underline{x}_1 | \underline{y} \rangle &= x_1^* \underline{y} = 0 \\ \langle \underline{x}_2 | \underline{y} \rangle &= x_2^* \underline{y} = 0 \end{aligned}\right\} \implies A^* \underline{y} = \underline{0} \quad \text{WHERE}$$

$$A = (\underline{x}_1, \underline{x}_2) \in M_{3,2}(\mathbb{C}) = \begin{pmatrix} i & 3 \\ 2 & 2-i \\ 1+i & i \end{pmatrix}.$$

- $\{\underline{b}_1, \dots, \underline{b}_n\}$ ORTHOGONAL IF $\underline{b}_i \perp \underline{b}_j \ \forall i \neq j$.

  $\{\underline{b}_1, \dots, \underline{b}_n\}$ ORTHONORMAL IF " " AND $\|\underline{b}_i\| = 1 \ \forall i = 1, \dots, n$.

THM. (i) (PYTHAGOREAN THM.)

$$\{\underline{b}_i\} \text{ ORTHOGONAL} \implies \|\underline{b}_1 + \dots + \underline{b}_n\|^2 = \|\underline{b}_1\|^2 + \dots + \|\underline{b}_n\|^2.$$

PF. $(n=2)$

$$\begin{aligned} \|\underline{x}+\underline{y}\|^2 &= \langle \underline{x}+\underline{y} | \underline{x}+\underline{y} \rangle \\ &= \langle \underline{x} | \underline{x} \rangle + \cancel{\langle \underline{x} | \underline{y} \rangle}^{\,0} + \cancel{\langle \underline{y} | \underline{x} \rangle}^{\,0} + \langle \underline{y} | \underline{y} \rangle \\ &= \|\underline{x}\|^2 + \|\underline{y}\|^2. \end{aligned}$$

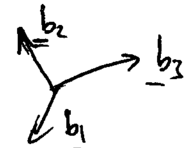
(ii) $\{\underline{b}_i\}$ ORTHOGONAL $\implies \{\underline{b}_i\}_{i=1}^n$ ARE LINEARLY INDEP.

PF.

$$\begin{aligned} c_1 \underline{b}_1 + \dots + c_n \underline{b}_n = \underline{0} &\implies \|\underline{0}\|^2 = |c_1|^2 \|\underline{b}_1\|^2 + \dots + |c_n|^2 \|\underline{b}_n\|^2 \\ &\implies c_i = 0 \ \forall i = 1, \dots, n. \\ &\implies \text{LINEARLY INDEP.} \end{aligned}$$

EXPANSION IN ORTHOGONAL BASIS:

SUPPOSE $\mathcal{B} = \{\underline{b}_1, \dots, \underline{b}_n\}$ IS ORTHOGONAL AND IS A BASIS OF V.

(

)

$$\underline{x} = a_1 \underline{b}_1 + \dots + a_n \underline{b}_n \iff [\underline{x}]_{\mathcal{B}} = (a_1, \dots, a_n)^T.$$

Q: HOW TO FIND a_i?

A:
$$\begin{aligned} \langle \underline{b}_i | \underline{x} \rangle &= \langle \underline{b}_i | a_1 \underline{b}_1 + \dots + a_n \underline{b}_n \rangle \\ &= a_1 \langle \underline{b}_i | \underline{b}_1 \rangle + \dots + a_n \langle \underline{b}_i | \underline{b}_n \rangle \\ &= a_i \langle \underline{b}_i | \underline{b}_i \rangle = a_i \|\underline{b}_i\|^2 \end{aligned}$$

$$\Rightarrow \boxed{a_i = \frac{\langle \underline{b}_i | \underline{x} \rangle}{\|\underline{b}_i\|^2}.}$$

THEREFORE,

$$\underline{x} = \sum_{i=1}^{n} \frac{\langle \underline{b}_i | \underline{x} \rangle}{\|\underline{b}_i\|^2} \underline{b}_i. \qquad \left(\underline{x} = \sum_{i=1}^{n} \langle \underline{b}_i | \underline{x} \rangle \underline{b}_i \quad \text{IF } \{\underline{b}_i\} \text{ ORTHONORMAL} \right)$$

NOTE: IN BRA-KET NOTATION, THIS IS

$$|\underline{x}\rangle = \sum_{i=1}^{n} \frac{\langle \underline{b}_i | \underline{x} \rangle}{\langle \underline{b}_i | \underline{b}_i \rangle} |\underline{b}_i\rangle = \left(\sum_{i=1}^{n} \frac{|\underline{b}_i\rangle \langle \underline{b}_i|}{\langle \underline{b}_i | \underline{b}_i \rangle} \right) |\underline{x}\rangle,$$

SO $$I = \sum_{i=1}^{n} \frac{|\underline{b}_i\rangle \langle \underline{b}_i|}{\langle \underline{b}_i | \underline{b}_i \rangle}$$ IS THE IDENTITY OPERATOR!

LECTURE 26
03/28/12

LAST TIME WE SAW THAT (IN BRA-KET NOTATION):

$$|\underline{x}\rangle = \underbrace{\left(\sum_{i=1}^{n} \frac{|\underline{b}_i\rangle\langle\underline{b}_i|}{\langle\underline{b}_i|\underline{b}_i\rangle} \right)}_{\underline{I}} |\underline{x}\rangle$$

WHAT IS $P_{\underline{b}_i} = \dfrac{|\underline{b}_i\rangle\langle\underline{b}_i|}{\langle\underline{b}_i|\underline{b}_i\rangle}$?

PROJECTIONS AND GRAM-SCHMIDT PROCESS (6.5-6.6):

DEF. $\underline{v} \in V$, $\underline{v} \neq \underline{0}$. LET $P_{\underline{v}} \doteq \dfrac{|\underline{v}\rangle\langle\underline{v}|}{\langle\underline{v}|\underline{v}\rangle}$

BE THE PROJECTION OPERATOR IN DIRECTION OF $\underline{v}$.

THM. FOR ANY $\underline{x} \in V$, CAN WRITE $\underline{x} = \underline{w} + \underline{y}$

WHERE FOR SOME GIVEN $\underline{v} \neq 0$, $\underline{w} \parallel \underline{v}$ AND $\underline{y} \perp \underline{v}$.

PR.
$$\underline{x} = \underline{I}\,\underline{x} = \left[P_{\underline{v}} + (\underline{I} - P_{\underline{v}})\right]\underline{x}$$
$$= \underbrace{P_{\underline{v}}\,\underline{x}}_{\text{CALL } \underline{w}} + \underbrace{(\underline{I} - P_{\underline{v}})\,\underline{x}}_{\text{CALL } \underline{y}}$$

~~LET~~ $W \subset V$ SUBSPACE WITH ORTHOGONAL BASIS $\{d_1, \ldots, d_m\}$.

DEF. $W^\perp \doteq$ SPACE OF ALL VECTORS $\perp$ TO W.

- NOTE THAT $\underline{x} \in W^\perp \iff P_W \underline{x} = \underline{0}$

$$\iff \underline{x} \in \mathrm{Ker}(P_W)$$

WHERE

$$P_W = \sum_{i=1}^{m} P_{\underline{d}_i} = \sum_{i=1}^{m} \frac{|\underline{d}_i\rangle\langle \underline{d}_i|}{\langle \underline{d}_i | \underline{d}_i\rangle}$$

IS THE PROJECTION ONTO SUBSPACE W.

- ANY $\underline{x} \in V$ IS $\underline{x} = \underline{w} + \underline{y}$ WHERE $\underline{w} \in W$, $\underline{y} \in W^\perp$.

SINCE

$$\underline{x} = P_W \underline{x} + \underbrace{(I - P_W)}_{P_W^\perp \doteq P_{W^\perp}} \underline{x}$$

EX. $V = \mathbb{R}^2$, $W = \text{SPAN}\{\underbrace{(1,1)^T}_{\text{CALL THIS } \underline{b}}\}$.

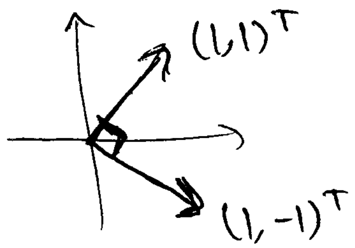
WHAT IS $W^\perp$?

$W^\perp =$ ALL $\underline{x} \in V$ S.T. $P_W \underline{x} = \underline{0}$.

$$P_W = \frac{|\underline{b}\rangle\langle \underline{b}|}{\langle \underline{b}|\underline{b}\rangle} = \frac{(1,1)^T(1,1)}{\|(1,1)^T\|^2} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

$\Rightarrow \ \mathrm{Ker}(P_W) = \text{span}\{(1,-1)^T\}.$

THIS MAKES SENSE, AS WE SEE BY A PICTURE:



<u>Ex.</u> $V = \mathbb{C}^2, \quad W = \text{span}\{\underbrace{(1-i, 2i)}_{\text{call this } \underline{b}}{}^T\}.$

$$P_W = \frac{(1-i, 2i)^T (1+i, -2i)}{\|(1-i, 2i)\|^2}$$

$$= \frac{1}{6}\begin{pmatrix} 2 & -2-2i \\ -2+2i & 4 \end{pmatrix}.$$

NOW FIND $\mathrm{Ker}(P_W)$ TO OBTAIN $W^\perp$.

<u>Q</u>: IN ORDER TO USE THE FORMULA

$$\underline{x} = \sum_{i=1}^{n} \frac{\langle \underline{d}_i | \underline{x} \rangle}{\|\underline{d}_i\|^2} \underline{d}_i,$$ WE HAD TO ASSUME

THAT $\{\underline{d}_i\}$ WERE ORTHOGONAL.

GIVEN SOME BASIS $\{\underline{b}_i\}$ OF V, HOW TO FIND AN ORTHOGONAL (OR ORTHONORMAL) BASIS OF V?

A: GRAM-SCHMIDT PROCEDURE.

IDEA: GIVEN BASIS $\mathcal{B} = \{\underline{b}_i\}_{i=1}^n$ OF V, START WITH FIRST VECTOR AND ITERATIVELY FIND "NEW" DIRECTIONS.

PROCEDURE: DEFINE NEW BASIS $\{\underline{d}_i\}_{i=1}^n$ BY:

1) $\underline{d}_1 = \underline{b}_1$

2) $$\underline{d}_2 = \underbrace{\left(I - P_{\underline{d}_1} \right)}_{P^{\perp}_{\underline{d}_1}} \underline{b}_2$$

3) $$\underline{d}_3 = \underbrace{\left(I - P_{\underline{d}_1} - P_{\underline{d}_2} \right)}_{P^{\perp}_{\mathrm{SPAN}\{\underline{d}_1, \underline{d}_2\}}} \underline{b}_3$$

$\vdots$

n) $$\underline{d}_n = \underbrace{\left(I - \sum_{i=1}^{n-1} P_{\underline{d}_i} \right)}_{P^{\perp}_{\mathrm{SPAN}\{\underline{d}_1, \dots, \underline{d}_{n-1}\}}} \underline{b}_n \, .$$

$\Rightarrow \mathcal{D} = \{\underline{d}_i\}_{i=1}^n$ <u>ORTHOGONAL</u> BASIS OF V!

IF WE LET $\underline{e}_i = \dfrac{\underline{d}_i}{\|\underline{d}_i\|}$ FOR EACH $i = 1, \dots, n$,

THEN $\mathcal{E} = \{\underline{e}_i\}_{i=1}^{n}$ IS AN <u>ORTHONORMAL</u> BASIS OF V.

<u>EX.</u> $V = \mathbb{R}^3$, $\mathcal{B} = \{(1,1,0)^T, (3,1,1)^T, (1,1,3)^T\}$.

TO FIND AN ORTHONORMAL BASIS OF $\mathbb{R}^3$, WE USE THE GRAM-SCHMIDT PROCEDURE ON $\mathcal{B}$:

1) $\underline{d}_1 = \underline{b}_1 = (1,1,0)^T$.

2) $\underline{d}_2 = (I - P_{\underline{d}_1})\underline{b}_2 = \underline{b}_2 - P_{\underline{d}_1}\underline{b}_2$

$$= (3,1,1)^T - \frac{\langle (1,1,0)^T \mid (3,1,1)^T \rangle}{\|(1,1,0)^T\|}(1,1,0)^T$$

$$= (1,-1,1)^T.$$

3) $\underline{d}_3 = (I - P_{\underline{d}_1} - P_{\underline{d}_2})\underline{b}_3 = \underline{b}_3 - P_{\underline{d}_1}\underline{b}_3 - P_{\underline{d}_2}\underline{b}_3$

$$= \underline{b}_3 - \underbrace{\frac{\langle \underline{d}_1 \mid \underline{b}_3 \rangle}{\|\underline{d}_1\|^2}}_{=1}\underline{d}_1 - \underbrace{\frac{\langle \underline{d}_2 \mid \underline{b}_3 \rangle}{\|\underline{d}_2\|^2}}_{=1}\underline{d}_2$$

$$= (-1,1,2)^T.$$

$\Rightarrow \mathcal{D} = \{(1,1,0)^T, (1,-1,1)^T, (-1,1,2)^T\}$. <u>ORTHOGONAL</u> BASIS.

$$\Rightarrow \quad \mathcal{E} = \left\{ \frac{1}{\sqrt{2}}(1,1,0)^T, \ \frac{1}{\sqrt{3}}(1,-1,1)^T, \ \frac{1}{\sqrt{6}}(-1,1,2)^T \right\}$$

ORTHONORMAL BASIS.

REMARK: IN GRAM SCHMIDT PROCEDURE, REMEMBER THAT

$$\underline{d}_i = \left(I - P_{d_1} - \dots - P_{d_{i-1}} \right) \underline{b}_i \qquad \checkmark$$

AND NOT

$$\cancel{\underline{d}_i = \left(I - P_{b_1} - \dots - P_{b_{i-1}} \right) \underline{b}_i}.$$

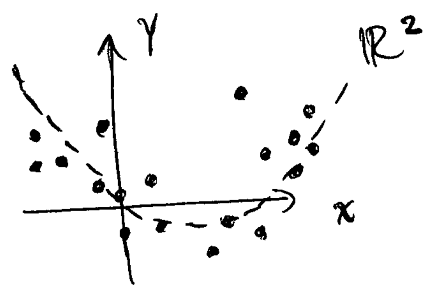
THAT IS, WE MUST USE THE ITERATIVE PROCEDURE TO OBTAIN A CORRECT ANSWER.

LECTURE 27
03/30/12

$A\underline{x} = \underline{b}$ HAS SOL'N $\underline{x} \iff \underline{b} \in \mathrm{Ran}(A)$
(COLUMN SPACE OF A).

<u>Q</u>: WHAT IF $\underline{b} \notin \mathrm{Ran}(A)$? CAN WE FIND $\underline{x}$ THAT "ALMOST" SOLVES $A\underline{x} = \underline{b}$? WHAT DO WE MEAN BY "ALMOST"?

<u>MOTIVATION</u>: FITTING CURVES TO DATA.



HAVE DATA $\{(x_i, y_i)\}_{i=1}^{m}$.
SUPPOSE WE EXPECT THE OBSERVED DATA TO FIT THE MODEL $y = cx + dx^2$.
UNKNOWN.

WHAT CHOICE FOR c, d? ACCORDING TO MODEL,

$$\begin{aligned} cx_1 + dx_1^2 &= y_1 \\ cx_2 + dx_2^2 &= y_2 \\ &\vdots \\ cx_m + dx_m^2 &= y_m \end{aligned} \iff \underbrace{\begin{pmatrix} x_1 & x_1^2 \\ \vdots & \vdots \\ x_m & x_m^2 \end{pmatrix}}_{A \in M_{m,2}(\mathbb{R})} \overbrace{\begin{pmatrix} c \\ d \end{pmatrix}}^{\underline{x} \in \mathbb{R}^2} = \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}}_{\underline{b} \in \mathbb{R}^m}.$$

LEAST SQUARES (6.7):

DEFINE ERROR $E(\underline{x}) = \| A\underline{x} - \underline{b} \|^2$.

GIVEN

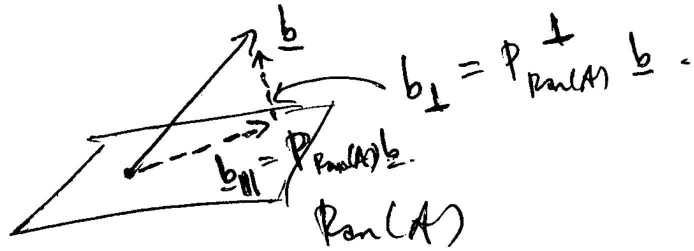
NOTE THAT $E(\underline{x}) = 0$ IFF $\underline{x}$ EXACTLY SOLVES $A\underline{x} = \underline{b}$!

DEF. $\underline{x}$ IS A LEAST SQUARES SOL'N TO $A\underline{x} = \underline{b}$

IF $E(\underline{x})$ IS MINIMIZED AT $\underline{x}$.

Q: GIVEN $A, \underline{b}$, HOW TO FIND $\underline{x}$? IS IT UNIQUE?

A:



$$\Rightarrow \underline{b} = \underline{b}_\perp + \underline{b}_\parallel$$

BY PYTHAGOREAN THM.,

$$E(\underline{x}) = \| \underline{b} - A\underline{x} \|^2 = \| \underbrace{\underline{b}_\perp}_{\in (Ran(A))^\perp} + \underbrace{(\underline{b}_\parallel - A\underline{x})}_{\in Ran(A)} \|^2$$

$$= \| \underline{b}_\perp \|^2 + \underbrace{\| \underline{b}_\parallel - A\underline{x} \|^2}_{\text{WANT TO MINIMIZE THIS.}}$$

SINCE $\underline{b}_\parallel \in Ran(A)$, THERE IS AT LEAST ONE SOL'N. $\underline{x}$ TO $A\underline{x} = \underline{b}_\parallel$, SO $\min_{\underline{x}} E(\underline{x}) = \| \underline{b}_\perp \|^2$.

SO, $\underline{x}$ IS A LEAST SQUARES SOL'N IF AND ONLY IF

$$A\underline{x} = b_{\parallel} = P_{\mathrm{Ran}(A)}\,\underline{b}.$$

GIVEN

<u>NOTE:</u> LEAST SQUARES SOLN. IS <u>UNIQUE</u> IF AND ONLY IF A HAS FULL RANK (I.E., RANK OF A EQUALS NUMBER OF COLUMNS).

<u>REMARK:</u> IT IS USUALLY NOT STRAIGHTFORWARD TO COMPUTE $P_{\mathrm{Ran}(A)}$. INSTEAD, WE USE ANOTHER APPROACH.

FIRST, NOTE THAT

$$\underline{b}_\perp \perp \mathrm{Ran}(A) \iff \underline{b}_\perp \perp \text{ columns of } A$$
$$\iff A^*\underline{b}_\perp = \underline{0}.$$

THEN, FOR ANY LEAST SQUARES SOL'N $\underline{x}$,

$$A^*(A\underline{x}) = A^*(\underline{b}_{\parallel}) = A^*(\underline{b} - \underline{b}_\perp) = A^*\underline{b}$$

<u>DEF.</u> (NORMAL EQN.) $(A^*A)\,\underline{x} = A^*\underline{b}$, $\quad A, \underline{b}$ GIVEN.

<u>THM.</u>
- $\underline{x}$ SOL'N. TO NORMAL EQN. $\iff$ $\underline{x}$ LEAST SQUARES SOL'N.
- $\underline{x}$ UNIQUE $\iff$ A HAS RANK $n = \#$ OF COLUMNS.

Ex. FIND BEST LINE THROUGH POINTS

$$(0,-1),\ (1,1),\ (2,4),\ (3,9).$$

THAT IS, WE SEEK CONSTANTS c, d FOR $y = c + dx$.

WRITING THE EQN'S $y_i = c + dx_i$ FOR $i = 1, \ldots, 4$.

IN MATRIX FORM,

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c \\ d \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} -1 \\ 1 \\ 4 \\ 9 \end{pmatrix}}_{\underline{b}}$$

$$A^*A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}$$

$$A^*\underline{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 13 \\ 36 \end{pmatrix}$$

__NORMAL EQN:__ $$\underbrace{\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix}}_{A^*A} \underbrace{\begin{pmatrix} c \\ d \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 13 \\ 36 \end{pmatrix}}_{\underline{b}}$$

$$\Rightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \frac{1}{10} \begin{pmatrix} -17 \\ 33 \end{pmatrix}.$$ (UNIQUE SOLN. SINCE A HAS FULL RANK.)

$\Rightarrow$ $y = -1.7 + 3.3\,x$ LINE OF BEST FIT.
(REGRESSION)

<u>NOTE:</u> NEED LINEAR EQN. FOR UNKNOWN PARAMETERS $c_0, c_1, \ldots, c_n$. MODEL FOR DATA DOESN'T NEED TO BE LINEAR!

<u>EX.</u> FIT DATA TO $\underbrace{y = c_0 + c_1 x}_{\text{LINEAR IN } c_i\text{'s.}}$. ✓

<u>EX</u> FIT DATA TO $\underbrace{c_0 y^2 - c_1 x^2 = c_2}_{\text{LINEAR IN } c_i\text{'s.}}$ ✓

<u>EX.</u> FIT DATA TO $\underbrace{y = c_0 e^{c_1 x}}_{\text{NOT LINEAR IN } c_i\text{'s}}$ ✗.

Lecture 28
04/02/12

Suppose V is an inner product space and $L : V \to V$ is a linear operator which is diagonalizable.

$\Rightarrow$ V has basis $\mathcal{B}$ of eigenvectors of L.

Q: Is $\mathcal{B}$ an orthogonal basis?

A: Generally not.

Ex. $V = \mathbb{R}^2$ w/ std. inner product.

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \in M_{2,2}(\mathbb{R}) \quad (\text{i.e., } A : V \to V).$$

E-values of A are $\lambda = 0$ and 3, with e-vectors $\xi_1 = (2,-1)^T$ and $\xi_2 = (1,1)^T$.

However, $\langle \xi_1 | \xi_2 \rangle = (2,-1)(1,1)^T = 1 \neq 0$, so $\xi_1 \not\perp \xi_2$.

Q: Given an inner product space, which class of operators generate an orthogonal basis of e-vectors? What is geometric structure of these operators?

ADJOINTS (7.1):

DEF. V INNER PROD. SPACE, $L : V \to V$.

THE ADJOINT OPERATOR L^* OF L IS THE UNIQUE OPERATOR THAT SATISFIES

$$\langle L^* \underline{x} \mid \underline{y} \rangle = \langle \underline{x} \mid L \underline{y} \rangle \quad \text{FOR ALL } \underline{x}, \underline{y} \in V.$$

- IF $V = \mathbb{C}^n$ w/ STD. INNER PRODUCT, THE ADJOINT OF $A \in M_{n,n}(\mathbb{C})$ IS $A^* = \overline{A}^T$.

- FOR A GENERAL VECTOR SPACE w/ ORTHONORMAL BASIS $\mathcal{E} = \{\underline{e}_i\}_{i=1}^n$:

MATRIX REPRESENTATION OF $L : V \to V$ IS

$$([L]_{\mathcal{E}})_{ij} = \langle \underline{e}_i \mid L \underline{e}_j \rangle$$

(EASY TO CHECK THIS, AND THAT $L = \sum_{i=1}^n \sum_{j=1}^n ([L]_{\mathcal{E}})_{ij} \, |\underline{e}_i\rangle\langle \underline{e}_j|$).

THEN,

$$\begin{aligned} ([L^*]_{\mathcal{E}})_{ij} = \langle \underline{e}_i \mid L^* \underline{e}_j \rangle &= \overline{\langle L^* \underline{e}_j \mid \underline{e}_i \rangle} \\ &= \overline{\langle \underline{e}_j \mid L \underline{e}_i \rangle} \\ &= \overline{([L]_{\mathcal{E}})_{ji}} \end{aligned}$$

$$\Rightarrow \quad [L^*]_{\mathcal{E}} = [L]_{\mathcal{E}}^{*} .$$

EX. $V = \mathbb{C}^3$, w/ STD. INNER PROD.

LET $L\underline{x} = (3x_1 + ix_2,\ ix_2 - 2x_3,\ (1+i)x_1 + 5x_3)^T$

WHERE $\underline{x} = (x_1, x_2, x_3)^T \in \mathbb{C}^3$.

Q: WHAT IS $L^*\underline{x}$?

A: $L = \begin{pmatrix} 3 & i & 0 \\ 0 & i & -2 \\ 1+i & 0 & 5 \end{pmatrix} \Rightarrow L^* = \overline{L}^T = \begin{pmatrix} 3 & 0 & 1-i \\ -i & -i & 0 \\ 0 & -2 & 5 \end{pmatrix}$

$$\Rightarrow L^*\underline{x} = (3x_1 + (1-i)x_3,\ -ix_1 - ix_2,\ -2x_2 + 5x_3)^T.$$

EX. $V = L^2(\mathbb{R})$ (INFINITE DIMENSIONAL SPACE OF SQUARE INTEGRABLE FCN'S)

I.E, $f \in L^2(\mathbb{R}) \Leftrightarrow \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$.

LET $\langle f | g \rangle = \int_{-\infty}^{\infty} f(x) g(x)\, dx$ BE THE INNER

PRODUCT (THIS IS WELL DEFINED SINCE

$$|\langle f|g\rangle| \leq \|f\|\,\|g\| = \left(\int_{-\infty}^{\infty} |f(x)|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |g(x)|^2 dx\right)^{1/2}$$

$< \infty$.)

IT IS EASY TO CHECK THAT $L = \frac{d}{dx}$ IS A LINEAR OPERATOR ON V (ASSUMING THAT ~~ALL~~ FCN'S IN V ARE DIFFERENTIABLE).

Q: WHAT IS L^*?

A:
$$\langle f | L g \rangle = \int_{-\infty}^{\infty} f(x) \left(\frac{d}{dx} g(x) \right) dx$$

INTEGRATION BY PARTS
$$= \underbrace{\Big[f(x)\, g(x) \Big]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \left(\frac{d}{dx} f(x) \right) g(x)\, dx$$

$= 0$ SINCE SQUARE INTEGRABILITY $\Rightarrow f(x), g(x) \to 0$ AS $x \to \pm\infty$

$$= \int_{-\infty}^{\infty} \left(-\frac{d}{dx} f(x) \right) g(x)\, dx$$

$$= \langle L^* f | g \rangle$$

SO, $L^* = -\frac{d}{dx}$.

- SIMILARLY, IF $\tilde{L}$ WERE $\frac{d^2}{dx^2}$, WE WOULD FIND THAT $\tilde{L}^* = \frac{d^2}{dx^2}$ BY INTEGRATING BY PARTS TWICE. IN THIS CASE, $\tilde{L}^* = \tilde{L}$ AND $\tilde{L}$ WOULD BE CALLED <u>SELF ADJOINT</u>.

PROPERTIES OF ADJOINTS :-

(i) $(A+B)^* = A^* + B^*$

(ii) $(AB)^* = B^* A^*$

(iii) $(cA)^* = \bar{c} A^*$

(iv) $(A^*)^* = A$.

THM. (i) $(\text{Ker } A^*) = (\text{Ran } A)^\perp$

(ii) $(\text{Ker } A) = (\text{Ran}(A^*))^\perp$

(iii) $(\text{Ker } A^*)^\perp = \text{Ran } A$

(iv) $(\text{Ker } A)^\perp = \text{Ran } A^*$.

PF. (i) $\underline{x} \in (\text{Ran } A)^\perp \iff \langle \underline{x} | A \underline{y} \rangle = 0$ FOR ALL $\underline{y}$

$\iff \langle A^* \underline{x} | \underline{y} \rangle = 0$ FOR ALL $\underline{y}$

$\iff A^* \underline{x} = 0$

$\iff \underline{x} \in \text{Ker } A^*$.

THEN, (i) $\iff$ (ii) $\iff$ (iv) $\iff$ (iii) AND WE ARE DONE.

LECTURE 29
04/04/12

DEF. AN OPERATOR $L : V \to V$ ON AN INNER PRODUCT SPACE V IS SELF-ADJOINT (OR HERMITIAN) IF $L^* = L$.

- IF V IS A REAL INNER PRODUCT SPACE, WE HAVE THAT $L^T = L$ AND L IS CALLED SYMMETRIC.

THM. EVERY EIGENVALUE OF A SELF-ADJOINT OPERATOR IS REAL.

PF. LET λ BE AN E-VALUE OF L WITH E-VECTOR $\underline{v}$. THEN,

$$\lambda \|\underline{v}\|^2 = \langle \underline{v} | \lambda \underline{v} \rangle = \langle \underline{v} | L\underline{v} \rangle = \langle L^* \underline{v} | \underline{v} \rangle$$
$$= \langle L\underline{v} | \underline{v} \rangle = \langle \lambda \underline{v} | \underline{v} \rangle = \bar{\lambda} \|\underline{v}\|^2.$$

$$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}.$$

THM. EIGENSPACES OF DISTINCT EIGENVALUES ARE ORTHOGONAL.

PF. LET λ, μ BE DISTINCT E-VALUES WITH CORRESPONDING EIGENSPACES E_λ AND E_μ. LET $\underline{v} \in E_\lambda$, $\underline{w} \in E_\mu$. THEN,

$$\langle L\underline{v} | \underline{w} \rangle = \bar{\lambda} \langle \underline{v} | \underline{w} \rangle = \lambda \langle \underline{v} | \underline{w} \rangle$$

SINCE $\lambda \in \mathbb{R}$.

$$\|$$

$$\langle \underline{v} | L\underline{w} \rangle = \mu \langle \underline{v} | \underline{w} \rangle$$

SO $\overbrace{(\lambda - \mu)}^{\neq 0} \langle \underline{v} | \underline{w} \rangle = 0$, AND $\langle \underline{v} | \underline{w} \rangle = 0 \Rightarrow \underline{v} \perp \underline{w}$.

NOTE: EIGENVECTORS CORRESPONDING TO SAME EIGENVALUE DON'T HAVE TO BE ORTHOGONAL !

EX.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \lambda = 1, \ -1$$

$$\Rightarrow \underbrace{\underline{\xi}_1 = (1,0,0)^T, \ \underline{\xi}_2 = (1,1,1)^T}_{\text{CORRESPONDING TO } \lambda = 1}, \ \underbrace{\underline{\xi}_3 = (0,1,-1)^T}_{\text{CORRESPONDING TO } \lambda = -1.}$$

EASY TO SEE $\underline{\xi}_3 \perp \text{SPAN}\{\underline{\xi}_1, \underline{\xi}_2\}$. ✓

BUT $\underline{\xi}_1 \not\perp \underline{\xi}_2$! HOWEVER, $\underline{\xi}_1$ AND $\underline{\xi}_2$ FORM A BASIS OF E_1, SO WE CAN ORTHOGONALIZE BY GRAM-SCHMIDT.

$$\Rightarrow \underline{b}_1 = \underline{\xi}_1 = (1,0,0)^T$$

$$\underline{b}_2 = (I - P_{\underline{b}_1})\underline{\xi}_2 = \underline{\xi}_2 - P_{\underline{b}_1}\underline{\xi}_2$$

$$= \underline{\xi}_2 - \frac{\langle \underline{b}_1 | \underline{\xi}_2 \rangle}{\|\underline{b}_1\|^2}\underline{b}_1$$

$$= (1,1,1)^T - \frac{1}{1}(1,0,0)^T$$

$$= (0,1,1)^T$$ ← STILL AN EIGENVECTOR CORRESPONDING TO $\lambda = 1$!

$$\Rightarrow \left.\begin{aligned} \underline{e}_1 &= \frac{\underline{b}_1}{\|\underline{b}_1\|} = (1,0,0)^T \\ \underline{e}_2 &= \frac{\underline{b}_2}{\|\underline{b}_2\|} = \frac{1}{\sqrt{2}}(0,1,1)^T \\ \underline{e}_3 &= \frac{\underline{\xi}_3}{\|\underline{\xi}_3\|} = \frac{1}{\sqrt{2}}(0,1,-1)^T \end{aligned}\right\} \text{ ORTHONORMAL BASIS OF EIGENVECTORS OF } A.$$

THM. LET L BE SELF-ADJOINT. ASSUME L IS DIAGONALIZABLE. THEN THERE IS AN ORTHONORMAL BASIS OF V CONSISTING OF EIGENVECTORS OF L.

PF. L HAS DISTINCT E-VALUES $\lambda_1, \dots, \lambda_r$, $r \leq n$, WITH EIGENSPACES $E_1, \dots, E_r$. THEN ALL THE E-SPACES ARE ORTHOGONAL. NOW USE GRAM-SCHMIDT TO FIND ORTHONORMAL BASIS $\mathcal{E}_i$ FOR EACH E_i. SINCE L IS DIAGONALIZABLE, $\mathcal{E} = \mathcal{E}_1 \cup \dots \cup \mathcal{E}_r$ IS A BASIS OF ORTHONORMAL E-VECTORS FOR V.

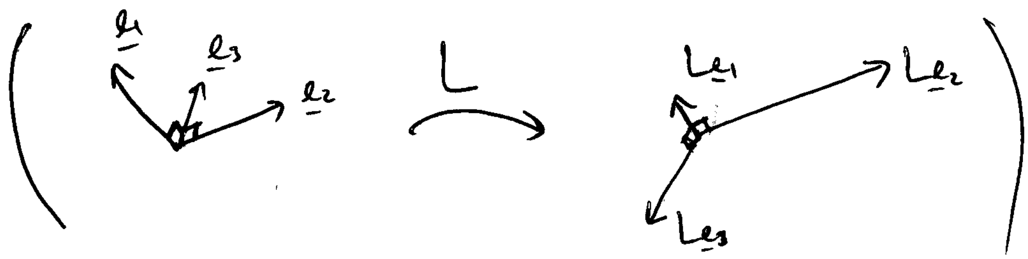
NOTE: WE WILL SHOW LATER THAT L SELF-ADJOINT IMPLIES L IS DIAGONALIZABLE, SO THE ASSUMPTION ABOVE CAN BE DROPPED.

THEREFORE,

THM. L SELF-ADJOINT $\iff$ $L = U D U^{-1}$

U: COLUMNS ARE ORTHONORMAL E-VECTORS OF L.

D: DIAGONAL MATRIX OF REAL EIGENVALUES OF L



PF. "⇒" ALREADY DONE.

"⇐" $L = UDU^{-1}$ WITH D REAL DIAGONAL

$U = \underbrace{(e_1, \dots, e_n)}_{\text{ORTHONORMAL BASIS } \mathcal{E}}$.

SO, $[L]_{\mathcal{E}} = D = D^* = [L]_{\mathcal{E}}^* = [L^*]_{\mathcal{E}}$,

AND $L^* = L$.

COROLLARY: (REAL SPECTRAL THM.)

IF V REAL INNER PRODUCT SPACE,

S SYMMETRIC $\iff$ $S = ODO^{-1}$

ORTHONORMAL E-VECTORS OF S.

DIAGONAL MATRIX OF REAL EIGENVALUES

Q: IS EVERY OPERATOR WHICH HAS AN ORTHONORMAL BASIS OF E-VECTORS SELF-ADJOINT?

A: NO. FOR EXAMPLE, ANY $L = UDU^{-1}$.

ORTHONORMAL E-VECTORS

DIAGONAL MATRIX W/ COMPLEX ENTRIES.

EX $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

$\Rightarrow \lambda_1 = 1 + 2i, \ \lambda_2 = 1 - 2i$ A NOT SELF-ADJOINT SINCE $\lambda \notin \mathbb{R}$

$e_1 = \frac{1}{\sqrt{2}}(1, i)^T, \ e_2 = \frac{1}{\sqrt{2}}(1, -i)^T$

$\Rightarrow \langle e_1 | e_2 \rangle = 0 \Rightarrow A = \underbrace{(e_1 \ e_2)}_{U} \overbrace{\begin{pmatrix} 1+2i & \\ & 1-2i \end{pmatrix}}^{D} \underbrace{(e_1 \ e_2)^{-1}}_{U^{-1}}$

Lecture 30
04/06/12

Last time, we saw that

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$ is <u>not</u> self-adjoint, but still has an orthonormal basis of eigenvectors.

<u>Note:</u>
$$A^*A = \begin{pmatrix} 1 & +2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ +2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$
$$\parallel$$
$$AA^* = \begin{pmatrix} 1 & -2 \\ +2 & 1 \end{pmatrix}\begin{pmatrix} 1 & +2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

<u>Normal operators.</u>

<u>Def.</u> $N : V \to V$ is <u>normal</u> if $N^*N = NN^*$ (i.e., N commutes with its adjoint).

<u>Note:</u> L self-adjoint $\Rightarrow$ L normal

<u>Pf:</u> $L = L^* \Rightarrow L^*L = LL^*$.

<u>Properties:</u>

- N normal $\Leftrightarrow$ $\|N\underline{x}\| = \|N^*\underline{x}\|$ for all $\underline{x} \in V$.

<u>Pf.</u> N normal $\Leftrightarrow$ $N^*N - NN^* = 0$

$$\Leftrightarrow \langle (N^*N - NN^*)\underline{x} \mid \underline{x} \rangle = 0 \quad \text{FOR ALL } \underline{x} \in V$$

(USING THAT $N^*N - NN^*$ IS SELF-ADJOINT AND THAT FOR ANY SELF-ADJOINT L, $\langle L\underline{x} \mid \underline{x} \rangle = 0$ FOR ALL $\underline{x} \in V \Leftrightarrow L = 0$).

$$\Leftrightarrow \langle N\underline{x} \mid N\underline{x} \rangle = \langle N^*\underline{x} \mid N^*\underline{x} \rangle$$

$$\Leftrightarrow \|N\underline{x}\| = \|N^*\underline{x}\|.$$

- SUPPOSE N NORMAL. THEN λ EIGENVALUE OF N WITH E-VECTOR $\underline{v}$ $\Leftrightarrow$ $\bar{\lambda}$ EIGENVALUE OF N^* WITH E-VECTOR $\underline{v}$.

<u>PF.</u>

$$0 = \|\underbrace{(N - \lambda I)}_{\text{NORMAL}}\underline{v}\| = \|(N - \lambda I)^* \underline{v}\|$$
$$= \|(N^* - \bar{\lambda} I)\underline{v}\|$$

$$\left(\text{SINCE } \begin{aligned} (N-\lambda I)^*(N-\lambda I) &= N^*N - \bar{\lambda}N - \lambda N^* + \lambda\bar{\lambda} I \\ &= NN^* - \bar{\lambda}N - \lambda N^* + \lambda\bar{\lambda} I \\ &= (N-\lambda I)(N-\lambda I)^*. \end{aligned} \right)$$

- N NORMAL $\Rightarrow$ EIGENSPACES OF DISTINCT EIGENVALUES ARE ORTHOGONAL.

<u>PF.</u> SUPPOSE λ, μ DISTINCT E-VALUES WITH CORRESPONDING EIGENSPACES E_λ AND E_μ. LET $\underline{v} \in E_\lambda$ AND $\underline{w} \in E_\mu$.

$$\left.\begin{array}{l} N\underline{v} = \lambda \underline{v} \\ N\underline{w} = \mu \underline{w} \end{array}\right\} \Rightarrow \overbrace{(\lambda - \mu)}^{\neq 0} \langle \underline{v} | \underline{w} \rangle$$

$$= \langle \bar{\lambda} \underline{v} | \underline{w} \rangle - \langle \underline{v} | \mu \underline{w} \rangle$$

$$= \langle N^* \underline{v} | \underline{w} \rangle - \langle \underline{v} | N \underline{w} \rangle$$

$$= \langle \underline{v} | N \underline{w} \rangle - \langle \underline{v} | N \underline{w} \rangle$$

$$= 0.$$

$$\Rightarrow \langle \underline{v} | \underline{w} \rangle = 0 \Rightarrow \underline{v} \perp \underline{w}.$$

BEFORE WE PROVE OUR MAIN RESULT, LET US INTRODUCE THE <u>SCHUR DECOMPOSITION</u> OF A MATRIX.

<u>THM.</u> (SCHUR DECOMPOSITION) ANY $A \in M_{n,n}(\mathbb{C})$ CAN BE WRITTEN AS $A = UTU^{-1}$.

ORTHONORMAL COLUMNS.

UPPER TRIANGULAR MATRIX

<u>PF.</u> (BY INDUCTION)

<u>CASE $n=1$</u>: $A \in \mathbb{C}$ ✓

<u>CASE $n-1$</u>: ASSUME TRUE.

<u>CASE n</u>: LET λ_1 BE E-VALUE OF A, W/ E-VECTOR $\underline{b}_1$, $\|\underline{b}_1\| = 1$. NOW LET $E = \underline{b}_1^{\perp}$ AND $\{\underline{e}_2, \ldots, \underline{e}_n\}$ ANY ORTHONORMAL BASIS OF E (TO BE CHOSEN LATER).

LET $\mathcal{E} = \{\underline{b}_1, \underline{e}_2, \ldots, \underline{e}_n\}$, SO $\mathcal{E}$ IS AN ORTHONORMAL BASIS OF $V = \mathbb{C}^n$.

$$\Rightarrow [A]_{\mathcal{E}} = \left(\begin{array}{c|c} \lambda_1 & ? \\ \hline 0 & \\ \vdots & A_1 \\ 0 & \end{array}\right)$$

WHERE $A_1 \in M_{n-1, n-1}(\mathbb{C})$.

BY INDUCTION HYPOTHESIS, A_1 IS TRIANGULAR, SO $[A]_{\mathcal{E}}$ IS TRIANGULAR AND $A = UTU^{-1}$

WITH $T = \left(\begin{array}{c|c} \lambda_1 & \\ \hline 0 & A_1 \end{array}\right)$ AND $U = (\underline{b}_1, \underline{e}_2, \ldots, \underline{e}_n)$.

FINALLY, WE USE THIS DECOMPOSITION TO SHOW:

<u>THM.</u> (COMPLEX SPECTRAL THM.)

$$N \text{ NORMAL} \iff N = U D U^{-1}$$

U: ORTHONORMAL BASIS OF E-VECTORS OF N.

D: DIAGONAL MATRIX OF E-VALUES OF N (POSSIBLY COMPLEX)

<u>Pf.</u> "$\Rightarrow$" N HAS SCHUR DECOMPOSITION

$$N = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{array}\right)$$

IN SOME ORTHONORMAL BASIS $\hat{\mathcal{E}}$, WHERE N_1 IS AN UPPER TRIANGULAR MATRIX.

$$(N^*N)_{11} = \bar{a}_{11} a_{11} = |a_{11}|^2.$$

$$\|$$

$$(NN^*)_{11} = a_{11}\bar{a}_{11} + \cdots + a_{1n}\bar{a}_{1n} = |a_{11}|^2 + \cdots + |a_{1n}|^2$$

$$\Rightarrow \quad a_{12} = \cdots = a_{1n} = 0.$$

SO, $N = \left(\begin{array}{c|c} a_{11} & 0 \\ \hline 0 & N_1 \end{array}\right)$ AND $N^*N = NN^*$

$$\Rightarrow \quad N_1^* N_1 = N_1 N_1^*.$$

REPEATING THE SAME STEPS SHOWS THAT $N = \left(\begin{array}{cc|c} a_{11} & & 0 \\ & a_{12} & \\ \hline 0 & & N_2 \end{array}\right)$,

SO THIS FINALLY GIVES THAT $N = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$

IN ORTHONORMAL BASIS $\mathcal{E}$.

"$\Leftarrow$" $N = UDU^{-1}$ WITH $U = (\underline{e}_1, \dots, \underline{e}_n)$

$\Rightarrow [N]_{\mathcal{E}} = D$ FOR $\mathcal{E} = \{\underline{e}_1, \dots, \underline{e}_n\}$.

$\Rightarrow [N^*]_{\mathcal{E}} = D^*$

THEN, $D^*D = DD^* \Rightarrow [N^*N]_{\mathcal{E}} = [NN^*]_{\mathcal{E}}$

$\Rightarrow N$ NORMAL.

LECTURE 31
04/09/12

RECALL THAT FOR A COMPLEX INNER PRODUCT SPACE V,

$$N \text{ NORMAL } \Longleftrightarrow N = UDU^{-1}$$
$(N^*N = NN^*)$

ORTHONORMAL E-VECTORS OF N — DIAGONAL MATRIX OF E-VALUES.

SPECIAL CASE:

$$L \text{ SELF-ADJOINT } \Longleftrightarrow L = UDU^{-1}$$
$(L^* = L)$

REAL DIAGONAL MATRIX OF E-VALUES

IF V REAL INNER PRODUCT SPACE,

$$S \text{ SYMMETRIC } \Longleftrightarrow S = ODO^{-1}$$
$(S^T = S)$

REAL MATRICES.

Q: WHAT ARE PROPERTIES OF U?

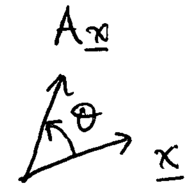
ISOMETRIES (7.4):

DEF. U IS AN ISOMETRY IF IT PRESERVES LENGTH — I.E.,

$$\|U\underline{x}\| = \|\underline{x}\| \quad \text{FOR ALL } x \in V.$$

NOTATION: IF V COMPLEX, U IS CALLED UNITARY.
" " REAL, " " " ORTHOGONAL.

Ex. $V = \mathbb{R}^2$, $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$\underline{x} = (x_1, x_2)^T$

$$\|A\underline{x}\|^2 = \|(cx_1 - sx_2, sx_1 + cx_2)^T\|^2$$

$$= c^2x_1^2 + s^2x_2^2 + s^2x_1^2 + c^2x_2^2$$

$$= x_1^2 + x_2^2$$

$$= \|\underline{x}\|^2, \quad \text{where } c = \cos\theta,\ s = \sin\theta$$

$\Rightarrow$ A ISOMETRY (I.E., A ORTHOGONAL MATRIX).

REMARK: IN FACT, FOR ANY REAL SPACE V, THERE IS AN ORTHONORMAL BASIS $\mathcal{E}$ OF V SUCH THAT FOR AN ORTHOGONAL MATRIX A,

$$[A]_{\mathcal{E}} = \begin{pmatrix} \boxed{B_1} & & & 0 \\ & \boxed{B_2} & & \\ & & \ddots & \\ 0 & & & \boxed{B_k} \end{pmatrix},$$ WHERE EACH BLOCK $\boxed{B_i}$

IS EITHER $\boxed{1}$, $\boxed{-1}$, OR $\boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$ FOR SOME $\theta \in (0, \pi)$.

UNCHANGED   REFLECTION   ROTATION IN PLANE

THM. THESE ARE ALL EQUIVALENT (I.E., (i)–(iv) AND (i′)–(iv′) ALL SAME):

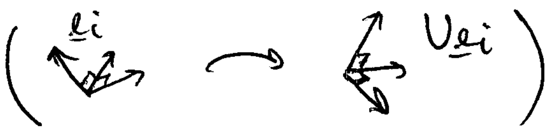
(i) U ISOMETRY	(i′) U^* ISOMETRY
(ii) $\langle U\underline{x} \mid U\underline{y}\rangle = \langle \underline{x} \mid \underline{y}\rangle$ FOR ALL $\underline{x}, \underline{y} \in V$	(ii′) $\langle U^*\underline{x} \mid U^*\underline{y}\rangle = \langle \underline{x} \mid \underline{y}\rangle$ FOR ALL $\underline{x}, \underline{y} \in V$

(iii) $U^*U = I$

(iv) IF $\mathcal{E} = \{\underline{e}_i\}$ ORTHONORMAL

$$[U]_{\mathcal{E}} = (U\underline{e}_1, \ldots, U\underline{e}_n)$$

HAS ORTHONORMAL COLUMNS



(iii') $UU^* = I$

(iv') IF $\mathcal{E} = \{\underline{e}_i\}$ ORTHONORMAL

$$[U^*]_{\mathcal{E}} = (U^*\underline{e}_1, \ldots, U^*\underline{e}_n)$$

HAS ORTHONORMAL COLUMNS.

IN ADDITION:

(a) U IS INVERTIBLE AND $U^{-1} = U^*$.

(PF. $\|U\underline{x}\| = \|\underline{x}\| \Rightarrow U\underline{x} = 0$ IFF $\underline{x} = 0$

$\Rightarrow \mathrm{Ker}(U) = \{0\} \Rightarrow U$ INVERTIBLE.

$U^*U = I \Rightarrow U^{-1} = U^*$.)

NOTE: THIS IMPLIES THAT THE SPECTRAL THM. CAN BE WRITTEN AS

N NORMAL $\Leftrightarrow N = UDU^*$ ← DIAGONAL.

(L SELF-ADJOINT $\Leftrightarrow L = UDU^*$ ← REAL DIAGONAL

S SYMMETRIC $\Leftrightarrow S = ODO^T$ ← REAL.)

(b) ALL E-VALUES OF U HAVE MAGNITUDE $|\lambda| = 1$

(SO IF U ORTHOGONAL, $\lambda = \pm 1$).

(PF. $\|U\underline{x}\| = \|\underline{x}\|$
$\|\lambda\underline{x}\| = |\lambda|\,\|\underline{x}\|$ $\Rightarrow |\lambda| = 1$.)

(c) $|\det U| = 1$

(PF. $|\det U| = |\lambda_1 \cdots \lambda_n| = 1$.)

PF. OF THM. :

(i) ⇒ (ii) : IF V REAL, $\langle \underline{x} | \underline{y} \rangle = \dfrac{\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2}{4}$ PARALLELOGRAM IDENTITY

$$\Rightarrow \langle U\underline{x} | U\underline{y} \rangle = \frac{\|U(\underline{x}+\underline{y})\|^2 - \|U(\underline{x}-\underline{y})\|^2}{4}$$

$$= \frac{\|\underline{x}+\underline{y}\|^2 - \|\underline{x}-\underline{y}\|^2}{4}$$

$$= \langle \underline{x} | \underline{y} \rangle .$$

SIMILAR IF V COMPLEX..

(ii) ⇒ (iii) : $\langle (U^*U - I)\underline{x} | \underline{y} \rangle = \langle U\underline{x} | U\underline{y} \rangle - \langle \underline{x} | \underline{y} \rangle$
$= 0$.

LET $\underline{y} = (U^*U - I)\underline{x} \Rightarrow \|(U^*U - I)\underline{x}\| = 0$ FOR ALL $\underline{x} \in V$

$\Rightarrow U^*U = I$.

(iii) ⇒ (iv) : $\langle U\underline{e}_i | U\underline{e}_j \rangle = \langle U^*U\underline{e}_i | \underline{e}_j \rangle = \langle \underline{e}_i | \underline{e}_j \rangle$
$= 0$.

(iv) ⇒ (i) : $\underline{x} = \sum_{i=1}^{n} x_i \underline{e}_i$

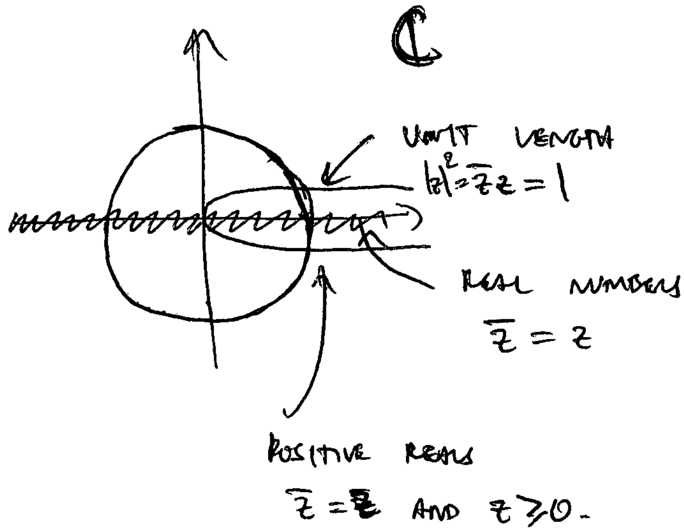
$\Rightarrow \|U\underline{x}\|^2 = \langle U\underline{x} | U\underline{x} \rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \langle U\underline{e}_i | U\underline{e}_j \rangle$

$$= x_1^2 + \cdots + x_n^2 = \|\underline{x}\|^2.$$

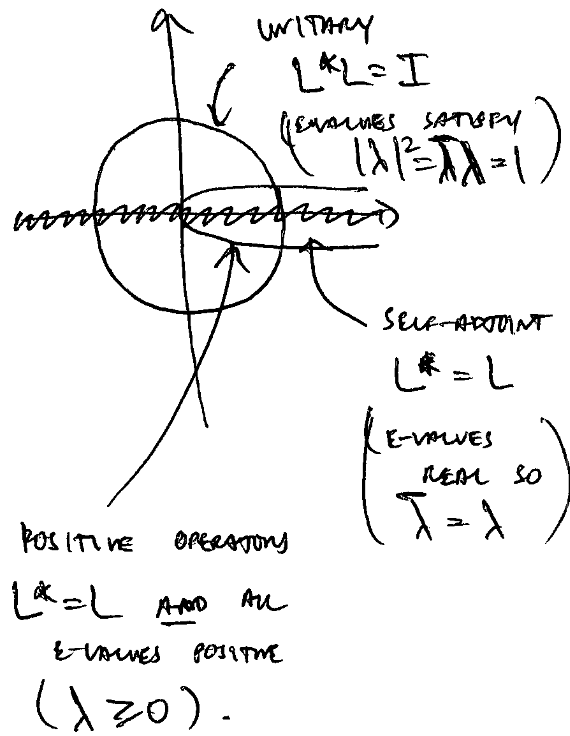
(i) $\Leftrightarrow$ (i''): EASY TO SHOW.

TO SUMMARIZE: THERE IS A NICE ANALOGY BETWEEN COMPLEX NUMBERS AND LINEAR OPERATORS!

COMPLEX NUMBERS



LINEAR OPERATORS



POLAR REPRESENTATION: FOR ANY z,

$$z = \frac{z}{|z|}|z| = \underbrace{\left(\frac{z}{|z|}\right)}_{\text{UNIT LENGTH}} \underbrace{\sqrt{\bar{z}z}}_{\text{POSITIVE REAL}}$$

POLAR DECOMPOSITION: FOR ANY L,

$$L = \underset{\substack{\uparrow \\ \text{UNITARY}}}{U} \underbrace{\sqrt{L^*L}}_{\text{POSITIVE OPERATOR}}.$$

LECTURE 32
04/11/12

POSITIVE OPERATORS:

DEF. L POSITIVE IF IT IS SELF-ADJOINT AND ALL ITS EIGENVALUES ARE ≥ 0. WE DENOTE THIS BY $L \geq 0$.

(NOTE: WE ALLOW EIGENVALUES TO BE ZERO.)

REMARK: L POSITIVE $\iff$ $L^* = L$ AND $\langle L\underline{x} | \underline{x} \rangle \geq 0$ FOR ALL $\underline{x} \in V$.

THM. IF $L \geq 0$, THERE IS A UNIQUE OPERATOR $B \geq 0$ SUCH THAT $B^2 = L$. IN PARTICULAR,

$$B = U D^{1/2} U^*, \quad \text{WHERE} \quad L = UDU^*$$

$$D^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

PF. $$B^2 = (UD^{1/2}\underbrace{U^*)(U}_{=I}D^{1/2}U^*) = UD^{1/2}D^{1/2}U^* = UDU^* = L.$$

UNIQUENESS EASY.

EX. SUPPOSE $A \in M_{2,2}(\mathbb{C})$ HAS E-VALUES $\lambda_1 = 9$, $\lambda_2 = 4$ W/ CORRESPONDING E-VECTORS $\underline{\xi}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$, $\underline{\xi}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$.

Q1: WHAT IS A?

A1: SINCE $\langle \underline{\xi}_1 | \underline{\xi}_2 \rangle = (-i, 1)(-i, 1)^T = 0$,

A HAS ORTHOGONAL E-VECTORS AND REAL E-VALUES

$\Rightarrow A$ SELF-ADJOINT.

$$\Rightarrow A = UDU^* = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 5i \\ -5i & 13 \end{pmatrix}.$$

Q2: WHAT IS $\sqrt{A}$?

A2: SINCE $A \geq 0$, $\sqrt{A}$ EXISTS.

$$\sqrt{A} = U \underbrace{D^{1/2}}_{= \begin{pmatrix} \sqrt{9} & 0 \\ 0 & \sqrt{4} \end{pmatrix}} U^* = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} = \begin{pmatrix} 5 & i \\ -i & 5 \end{pmatrix}.$$

NOTE: $\sqrt{A}$ IS DETERMINED BY E-VALUES OF A, NOT BY INDIVIDUAL ENTRIES a_{ij}!

WE WILL USE POSITIVE OPERATORS TO INTRODUCE ONE OF THE MOST USEFUL MATRIX DECOMPOSITIONS, CALLED SINGULAR VALUE DECOMPOSITION (SVD).

MOTIVATION: SO FAR WE HAVE INTRODUCED SEVERAL DECOMPOSITIONS.

A DIAGONALIZABLE

$$A = PDP^{-1}$$

P: E-VECTORS; D: DIAGONAL OF E-VALUES

$\rightsquigarrow$

GENERAL $A \in M_{n,n}(\mathbb{C})$

$$A = \tilde{P}\tilde{D}\tilde{P}^{-1}$$

$\tilde{P}$: E-VECTORS AND GENERALIZED E-VECTORS. $\tilde{D}$: BLOCK DIAGONAL

$\boxed{\text{JORDAN DECOMPOSITION}}$

A NORMAL

$$A = UDU^*$$

U: ORTHONORMAL E-VECTORS; D: DIAGONAL OF E-VALUES

$\rightsquigarrow$

GENERAL $A \in M_{n,n}(\mathbb{C})$

$$A = UTU^*$$

U: ORTHONORMAL BASIS; T: UPPER TRIANGULAR W/ E-VALUES ALONG DIAGONAL

$\boxed{\text{SCHUR DECOMPOSITION}}$

$\rightsquigarrow$

GENERAL $A \in M_{n,n}(\mathbb{C})$

$$A = U\Sigma V^*$$

U: UNITARY; Σ: DIAGONAL MATRIX OF SINGULAR VALUES. V^*: UNITARY

$\boxed{\text{SINGULAR VALUE DECOMPOSITION}}$

THAT IS,

Q: CAN WE FIND TWO ORTHONORMAL BASES $\mathcal{E}$ AND $\mathcal{F}$ SUCH THAT $[A]_{\mathcal{E}\mathcal{F}}$ IS DIAGONAL?

A: YES, FOR ANY A. THEN U WILL HAVE COLUMNS FROM $\mathcal{E}$, V COLUMNS FROM $\mathcal{F}$, AND $\Sigma = [A]_{\mathcal{E}\mathcal{F}}$.