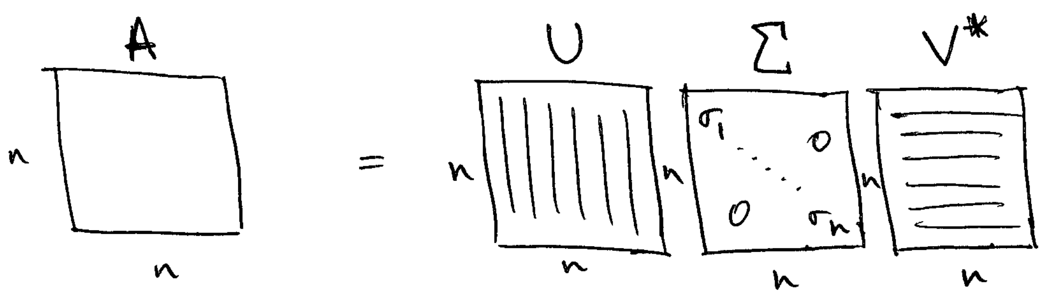
LECTURE 33
04/13/12

SINGULAR VALUE DECOMPOSITION (SVD):

FOR ANY $A \in M_{n,n}(\mathbb{C})$,



$$\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n), \quad \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$$

SINGULAR VALUES (E-VALUES OF $\sqrt{A^*A}$ OR $\sqrt{AA^*}$)

V UNITARY (ORTHONORMAL E-VECTORS OF $\sqrt{A^*A}$)

U UNITARY (ORTHONORMAL E-VECTORS OF $\sqrt{AA^*}$)

NOTE: A^*A AND AA^* ARE POSITIVE OPERATORS SINCE THEY ARE SELF-ADJOINT AND HAVE ALL EIGENVALUES POSITIVE (≥ 0). TO SEE THIS, NOTE THAT $(A^*A)^* = A^*A^{**} = A^*A$. AND IF λ IS AN E-VALUE W/ E-VECTOR $\underline{v}$ FOR A^*A, THEN

$$\lambda \|\underline{v}\|^2 = \lambda \langle \underline{v} | \underline{v} \rangle = \langle \underline{v} | \lambda \underline{v} \rangle = \langle \underline{v} | A^*A\underline{v} \rangle$$
$$= \langle A\underline{v} | A\underline{v} \rangle \quad \text{(BY PROPERTIES OF ADJOINT.)}$$
$$= \|A\underline{v}\|^2 \Rightarrow \lambda \geq 0.$$

Q: WHY SVD?

A: WE WILL SEE LATER THAT SVD IS USEFUL IN APPLICATIONS INVOLVING LARGE, HIGH-DIMENSIONAL DATA SETS. IT IS NOT AS USEFUL FOR DYNAMICAL APPLICATIONS SINCE

$$A^k = (U\Sigma V^*)^k = U\Sigma V^* U\Sigma V^* \cdots U\Sigma V^*$$
$$\neq U\Sigma^k V^*.$$

(UNLESS $U=V$, IN WHICH CASE THIS IS THE SAME AS JORDAN FORM.)

REMARK:

1) A^*A AND AA^* HAVE SAME E-VALUES!

PF. $A^*A\underline{v} = \lambda\underline{v} \Rightarrow AA^*(\underbrace{A\underline{v}}_{\text{CALL } \underline{u}}) = \lambda(\underbrace{A\underline{v}}_{\underline{u}})$

$AA^*\underline{u} = \lambda\underline{u} \Rightarrow A^*A(\underbrace{A^*\underline{u}}_{\text{CALL } \underline{v}}) = \lambda(\underbrace{A^*\underline{u}}_{\underline{v}}).$

2) $r \doteq \text{RANK}(A) = \#$ OF NONZERO σ_i's.

PF. $\text{RANK}(A) = \text{RANK}([A]_{\mathcal{EF}}) = \text{RANK}(\Sigma)$
$= \#$ NONZERO σ_i's

WHERE $\mathcal{E}, \mathcal{F}$ ARE COLUMNS OF U, V.

PROCEDURE TO FIND SVD:

① FIND NONZERO E-VALUES $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, $r \leq n$. OF A^*A AND CORRESPONDING E-VECTORS $\{v_i\}_{i=1}^r$.

② LET $\sigma_i = \sqrt{\lambda_i}$, $i = 1, .., r$. THESE ARE THE E-VALUES OF $\sqrt{A^*A}$ (OR OF $\sqrt{AA^*}$) — THAT IS, THE SINGULAR VALUES OF A.

③ LET $\underline{u}_i = \frac{1}{\sigma_i} A\underline{v}_i$, $i = 1, .., r$. THESE ARE THE ORTHONORMAL E-VECTORS OF AA^*.

(TO SEE THIS, NOTE THAT $\langle A\underline{v}_i | A\underline{v}_j \rangle = \langle A^*A\underline{v}_i | \underline{v}_j \rangle$

$$= \begin{cases} \sigma_i^2 & \text{IF } i = j \\ 0 & \text{ELSE} \end{cases} \Rightarrow \{\underline{u}_i\} \text{ ORTHONORMAL}.)$$

④ IF NECESSARY: FOR REMAINING COLUMNS

$\underline{v}_{r+1}, \ldots, \underline{v}_n$ (E-VECTORS OF A^*A W/ E-VALUE 0, I.E, IN $\mathrm{Ker}(A^*A)$)

$\underline{u}_{r+1}, \ldots, \underline{u}_n$ (E-VECTORS OF AA^* W/ E-VALUE 0, I.E, IN $\mathrm{Ker}(AA^*)$)

USE GRAM-SCHMIDT. THAT IS, FIND A BASIS FOR $\mathrm{Ker}(A^*A)$ AND ORTHONORMALIZE TO GET $\{v_i\}_{i=r+1}^n$, AND SIMILARLY FIND A BASIS FOR $\mathrm{Ker}(AA^*)$ AND ORTHONORMALIZE TO GET $\{\underline{u}_i\}_{i=r+1}^n$.

Ex. Find SVD of $A=\begin{pmatrix}2&2\\-1&1\end{pmatrix}$.

① $A^*A=\begin{pmatrix}2&-1\\2&1\end{pmatrix}\begin{pmatrix}2&2\\-1&1\end{pmatrix}=\begin{pmatrix}5&3\\3&5\end{pmatrix} \Rightarrow \lambda_1=8,\ \lambda_2=2$

(Trace $=10$, det $=16$)

$\Rightarrow \sqrt{\lambda_1}=\sqrt{8},\ \sqrt{\lambda_2}=\sqrt{2}$

② $\Rightarrow \sigma_1=\sqrt{8},\ \sigma_2=\sqrt{2}$.

Corresponding eigvectors of A^*A are $\underline{v}_1=\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix},\ \underline{v}_2=\frac{1}{\sqrt2}\begin{pmatrix}-1\\1\end{pmatrix}$

orthonormal.

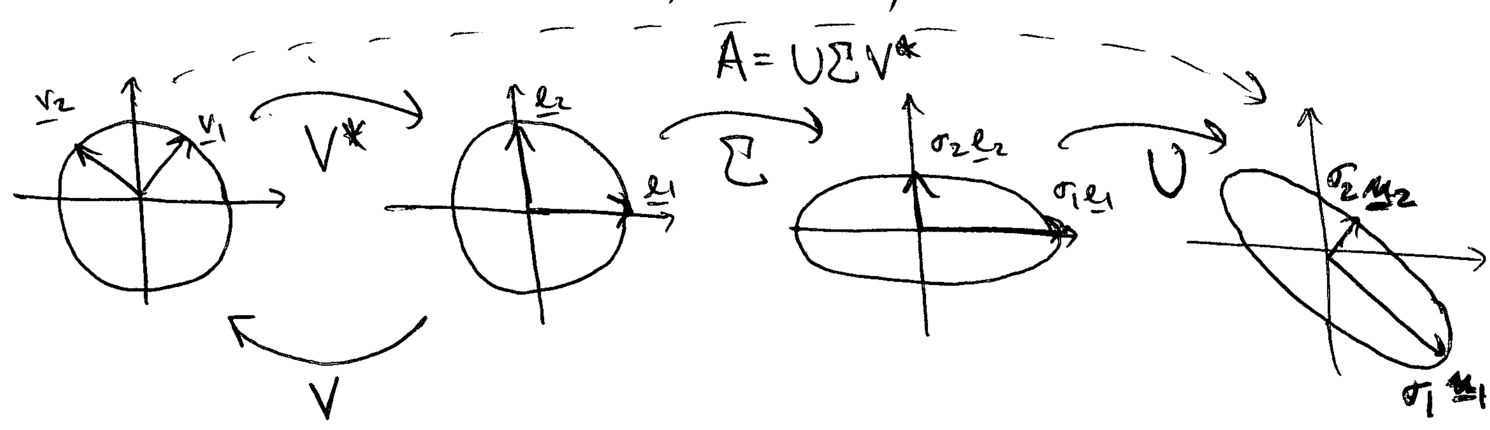
③ $\frac{1}{\sigma_1}A\underline{v}_1 \doteq \underline{u}_1=\begin{pmatrix}1\\0\end{pmatrix}$

$\frac{1}{\sigma_2}A\underline{v}_2 \doteq \underline{u}_2=\begin{pmatrix}0\\1\end{pmatrix}$

orthonormal.

$$\Rightarrow A = \underset{\begin{pmatrix}1&0\\0&1\end{pmatrix}}{\underset{\shortparallel}{U}}\ \underset{\begin{pmatrix}\sqrt8&0\\0&\sqrt2\end{pmatrix}}{\underset{\shortparallel}{\Sigma}}\ \underset{\frac{1}{\sqrt2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}}{\underset{\shortparallel}{V^*}}$$

Geometry of SVD: *Every* linear transformation is given by a rotation, stretch, and another rotation. For ex., in $\mathbb{R}^2$,



Q: WHY DOES A^*A PLAY A ROLE TO BEGIN WITH?

FROM THE GEOMETRIC PICTURE, WE SEE THAT ANY LINEAR TRANSFORMATION IS A MAPPING FROM THE UNIT SPHERE TO AN ELLIPSE (A HYPERELLIPSE IN HIGHER DIMENSIONS).

WE SEEK $\underline{x}$'s SUCH THAT $\|\underline{x}\|=1$ AND $A\underline{x}$ HAS MAXIMAL (OR MINIMAL) LENGTH.

Let $Q(\underline{x}) = \|A\underline{x}\|^2$. THEN, WE SEEK SOLUTIONS $\underline{x}$ THAT

$$\begin{cases} \text{EXTREMIZE } Q(\underline{x}) = \|A\underline{x}\|^2 = \langle A\underline{x} | A\underline{x}\rangle = \langle \underline{x} | A^*A\underline{x}\rangle \\ \text{w/ CONSTRAINT } \|\underline{x}\|=1. \end{cases}$$

USING LAGRANGE MULTIPLIERS TO SOLVE THIS CONSTRAINED MAXIMIZATION/MINIMIZATION PROBLEM, WE GET THAT SOLUTIONS SATISFY

$$A^*A\,\underline{x} = \lambda\,\underline{x} \quad \text{FOR SOME } \lambda$$

LAGRANGE MULTIPLIER

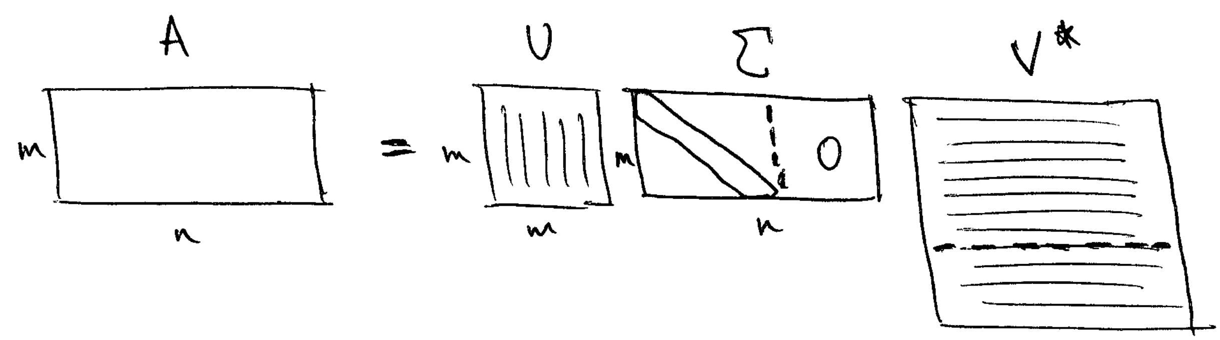
$\Rightarrow$ I.E., LOOK AT E-VECTORS OF A^*A!

LECTURE 34
04/16/12

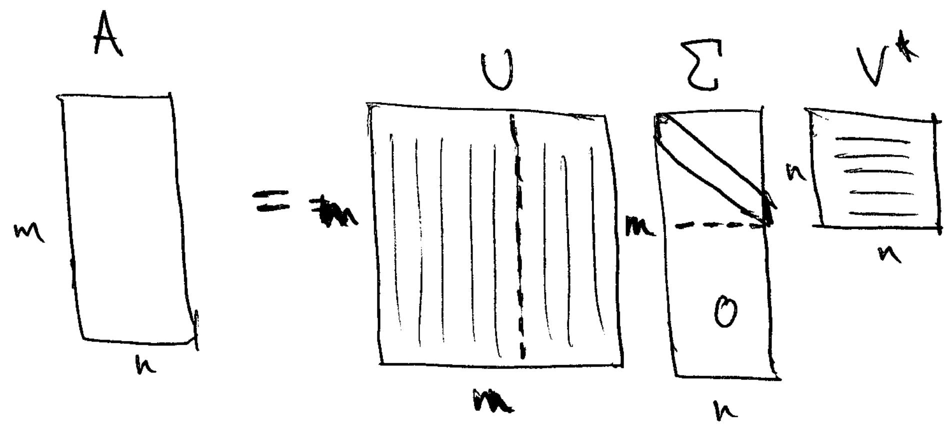
SINGULAR VALUE DECOMPOSITION (SVD) (CONT'D):

LAST TIME, WE INTRODUCED SVD FOR ANY SQUARE MATRICES —
IN FACT, CAN FIND SVD OF ANY $A \in M_{m,n}(\mathbb{C})$
m,n MAY BE DIFFERENT!

$(m<n)$



$(n<m)$



$\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r)$ (RECTANGULAR DIAGONAL MATRIX w/ SINGULAR VALUES ON DIAGONAL)

$V \in M_{n,n}(\mathbb{C})$ UNITARY (E-VECTORS OF A^*A)

$U \in M_{m,m}(\mathbb{C})$ UNITARY (E-VECTORS OF AA^*)

PROCEDURE TO FIND FIND SVD EXACTLY SAME AS BEFORE, EXCEPT

(4') IF NECESSARY: $\underline{v}_{r+1}, \dots, \underline{v}_{n} \in \mathrm{Ker}(A^*A)$

$\underline{u}_{r+1}, \dots, \underline{u}_{m} \in \mathrm{Ker}(AA^*)$

FIND USING GRAM-SCHMIDT.

Ex. FIND SVD OF $A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}$

(1) $A^*A = \begin{pmatrix} 9 & 9 \\ 8 & 8 \end{pmatrix} \Rightarrow \lambda_1 = 17, \ \lambda_2 = 1$

W/ CORRESPONDING E-VECTORS

$\underline{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \underline{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

ORTHONORMAL.

(2) $\sigma_1 = \sqrt{\lambda_1} = \sqrt{17}, \quad \sigma_2 = \sqrt{\lambda_2} = 1.$

(3) $\underline{u}_1 = \frac{1}{\sigma_1} A \underline{v}_1 = \frac{1}{\sqrt{17}\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{34}}\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}$

$\underline{u}_2 = \frac{1}{\sigma_2} A \underline{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 2 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

(4) STILL NEED $\underline{u}_3$. NOTE THAT

$AA^* = \begin{pmatrix} 5 & 6 & 4 \\ 6 & 8 & 6 \\ 4 & 6 & 5 \end{pmatrix} \xrightarrow{\text{ROW REDUCE}} \begin{pmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \operatorname{Ker}(AA^*) = \text{SPAN}\left\{\frac{1}{\sqrt{17}}(2,-3,2)^T\right\}$$

$$\Rightarrow \underline{u}_3 = \frac{1}{\sqrt{17}}(2,-3,2)^T .$$

(EASY TO CHECK THAT $\underline{u}_3 \perp \underline{u}_1$ AND $\underline{u}_3 \perp \underline{u}_2$).

So,

$$A = U \Sigma V^*$$

$$U = \begin{pmatrix} \frac{3}{\sqrt{34}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \\ \frac{4}{\sqrt{34}} & 0 & \frac{-3}{\sqrt{17}} \\ \frac{3}{\sqrt{34}} & \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{17}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{17} & 0 \\ 0 & 1 \end{pmatrix}, \quad V^* = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Applications of SVD:

IT IS EASY TO SEE THAT $A = U \Sigma V^*$

$$\Rightarrow A = \sum_{i=1}^{r} \sigma_i \underline{u}_i \underline{v}_i^* \qquad (r = \text{RANK}(A)).$$

NOTE THAT $\underline{u}_i \underline{v}_i^*$ IS A RANK 1 MATRIX (WHY?),

SO $A = \sum_{i=1}^{p} \sigma_i \underline{u}_i \underline{v}_i^*$ IS A RANK p MATRIX.

AS WE NOW SEE, THIS IS THE "BEST" p-RANK APPROXIMATION TO A IN SOME SENSE.

WE NEED A NOTION OF DISTANCE FOR MATRICES.

CONSIDER THE FROBENIUS INNER PRODUCT ON $V = M_{m,n}(\mathbb{C})$:

$$\langle A | B \rangle = \mathrm{Tr}(A^* B).$$

$$\Rightarrow \|A\|^2 = \langle A | A \rangle = \mathrm{Tr}(A^* A) = \sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2.$$

CAN CHECK THAT FROBENIUS NORM IS UNITARILY INVARIANT — I.E.,

$$W \text{ ISOMETRY} \Rightarrow \begin{array}{l} \|WA\| = \|A\| \\ \|AW\| = \|A\| \end{array} \text{ FOR ANY } A.$$

THM. IF $A = U \Sigma V^*$, THE CLOSEST (IN SQUARED FROBENIUS NORM!) p-RANK APPROXIMATION TO A IS $B = U \Sigma_p V^*$, WHERE $\Sigma_p = \mathrm{DIAG}(\sigma_1, \ldots, \sigma_p)$

PF. WE SEEK TO SOLVE

$$\begin{cases} \text{MINIMIZE } \|A - B\|^2 \text{ OVER ALL } B \text{ SUCH THAT} \\ \mathrm{RANK}(B) \leq p \leq r. \end{cases}$$

IF A HAS SVD $A = U \Sigma V^*$, DEFINE A NEW MATRIX $S = U^* B V$, SO THAT $B = U S V^*$.

THEN,

$$\begin{aligned} \|A - B\|^2 &= \|U \Sigma V^* - U S V^*\|^2 \\ &= \|U (\Sigma - S) V^*\|^2 \end{aligned}$$

$$= \|\Sigma - S\|^2 \quad (\text{since } U, V^* \text{ isometries})$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} |\Sigma_{ij} - S_{ij}|^2$$

$$= \sum_{i=1}^{r} |\sigma_i - S_{ii}|^2 + \sum_{\substack{i,j \text{ s.t.} \\ \Sigma_{ij}=0}}\sum |S_{ij}|^2 .$$

- TO MAKE THE 2ND TERM AS SMALL ~~AS~~ POSSIBLE, WE TAKE $S_{ij} = 0$ EVERYWHERE $\Sigma_{ij} = 0$. SO, AT THIS POINT WE KNOW THAT $S = \text{diag}(S_{11}, S_{22}, \ldots, S_{rr})$.
- TO MAKE THE 1ST TERM AS SMALL AS POSSIBLE, WE TAKE $S_{ii} = \sigma_i$ FOR $i = 1, \ldots, p$, AND ZERO OTHERWISE. THIS IS THE BEST S OF RANK $\leq p$ WHICH MINIMIZES $\|\Sigma - S\|^2$.

SO, $S = \text{diag}(\sigma_1, \ldots, \sigma_p) \Rightarrow B = U\Sigma_p V^*$ IS THE BEST p-RANK APPROX!

THIS PROPERTY HAS WIDE APPLICABILITY TO DATA COMPRESSION, PRINCIPAL COMPONENT ANALYSIS, FACTOR ANALYSIS, PATTERN RECOGNITION, ...

EX. $A = \begin{pmatrix} 1.01 & 1 & 1 \\ 1 & 1.01 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

RANK$(A) = 3$, BUT IT SHOULD BE VERY CLOSELY APPROX. BY A RANK 1 MATRIX SINCE $1.01 \approx 1$. WE FIND

$$\Sigma = \begin{pmatrix} \overset{\sigma_1}{\overset{\shortparallel}{3.01}} & & \\ & \overset{\sigma_2}{\overset{\shortparallel}{0.01}} & \\ & & \overset{\sigma_3}{\overset{\shortparallel}{0.01}} \end{pmatrix}$$ SO $\sigma_1 \gg \sigma_2, \sigma_3$.

$\rightsquigarrow$ RANK$(A) = 3$ BUT IS VERY "CLOSE" TO RANK 1 MATRIX

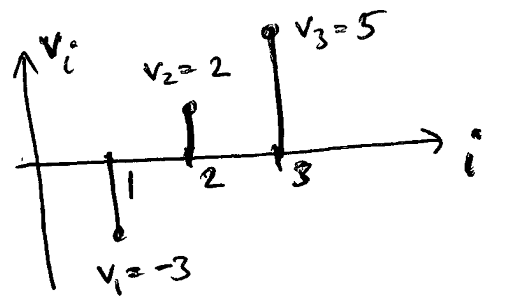
$$\sigma_1 \underline{u}_1 \underline{v}_1^* = \begin{pmatrix} 1.01 & 1.01 & 1.01 \\ 1.01 & 1.01 & 1.01 \\ 1.01 & 1.01 & 1.01 \end{pmatrix}$$

LECTURE 35
04/18/12

INFINITE-DIM VECTOR SPACES, INNER PROD. SPACES (6.8):

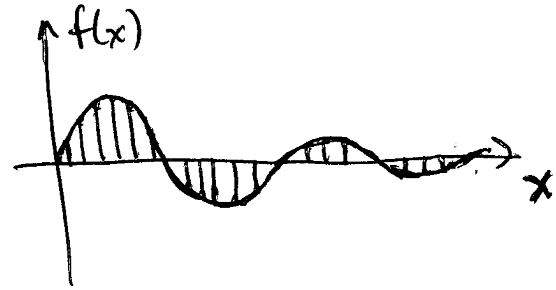
WE CAN THINK OF FUNCTIONS AS INF.-DIM. VECTORS, BY THE FOLLOWING ANALOGY:

$$\underline{v} = (v_1, v_2, v_3)^T = \{v_i\}_{i=1}^3 \in \mathbb{R}^3, \quad \text{FOR EX.,} \quad \underline{v} = (-3, 2, 5)^T$$



ANALOGOUSLY,

$$f = \{f(x)\}_{x \in \mathbb{R}} \in V, \quad \text{FOR EX.,} \quad f(x) = e^{-x} \sin x$$



<u>MOTIVATION</u>: INFINITE-DIM. SPACES ARISE IN MANY SETTINGS, BUT ARE ESPECIALLY IMPORTANT WHEN CONSIDERING PARTIAL DIFFERENTIAL EQUATIONS (PDE).

RECALL THAT FOR LINEAR, CONST-COEFF., HOMOGENEOUS ODE WE CONSIDERED

$$\begin{cases} \dfrac{d\underline{v}}{dt} = A\underline{v}, \\ \underline{v}(0) \text{ GIVEN} \end{cases} \quad \text{WHERE} \quad \underline{v} = \underline{v}(t) \in \mathbb{R}^n \text{ FOR ALL } t \geq 0.$$

AND $A : \mathbb{R}^n \to \mathbb{R}^n$ LINEAR OPERATOR.

Similarly, linear, const. coeff., homogeneous PDE take the form

$$\begin{cases} \dfrac{du}{dt} = Lu \\ u(t=0) \text{ given} \end{cases},$$

where $u = u(t) \in V$ for all $t \geq 0$ and $L : V \to V$ linear operator.

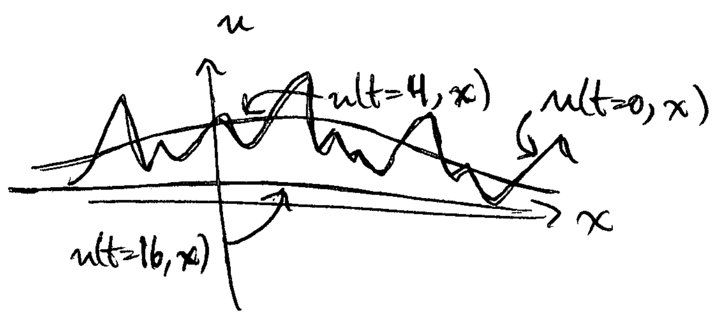
Here, V is a <u>function space</u> (i.e, a space in which the typical element is a function $f(x)$, $x \in \mathbb{R}$).

Therefore, we should think of $u(t) \doteq u(t, x)$ for $x \in \mathbb{R}$ and L as an operator on functions of x. That is,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = L\, u(x,t) \\ u(0, x) \text{ given} \end{cases}.$$

For ex., if $L = \dfrac{\partial^2}{\partial x^2}$ this is a classical PDE called the <u>heat eqn.</u>:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} \\ u(0, x) \text{ given} \end{cases} \longrightarrow$$



<u>Warning</u>: While many of the concepts we have learned for finite-dim. spaces will carry over to ∞-dim. spaces, many will not. Issues will typically arise due to a lack of convergence of some sum.

FOR EX., CONSIDER ANY TWO FINITE-DIM $n \times n$ MATRICES A_n AND B_n.

THEN $\mathrm{Tr}(A_nB_n) = \mathrm{Tr}(B_nA_n) \Rightarrow \mathrm{Tr}(A_nB_n - B_nA_n) = 0$

FOR ANY $A_n, B_n \in M_{n,n}(\mathbb{C})$.

HOWEVER, IF

$$A_\infty = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \vdots \\ & & \ddots & \ddots \\ 0 & \cdots & & \end{pmatrix}, \quad B_\infty = A_\infty^T = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \vdots \\ & 1 & \ddots & \\ 0 & \cdots & \ddots & \end{pmatrix}$$

INFINITE-DIM MATRICES.

THEN $$A_\infty B_\infty = \begin{pmatrix} 1 & & 0 \\ & 1 & \vdots \\ & & \ddots \\ 0 & \cdots & \end{pmatrix}, \quad B_\infty A_\infty = \begin{pmatrix} 0 & & 0 \\ & 1 & \vdots \\ & & 1 & \\ 0 & \cdots & & \ddots \end{pmatrix}$$

$$\Rightarrow A_\infty B_\infty - B_\infty A_\infty = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ & & 0 & \\ 0 & & & \ddots \end{pmatrix} \Rightarrow \mathrm{Tr}(A_\infty B_\infty - B_\infty A_\infty) = 1 \neq 0 \,!$$

WE SEE THAT $\lim\limits_{n\to\infty} \mathrm{Tr}(A_nB_n - B_nA_n) \neq \mathrm{Tr}\left(\lim\limits_{n\to\infty}(A_nB_n - B_nA_n)\right)$

WHERE $$A_n = \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}}_{n}\Big\}n, \quad B_n = A_n^T = \underbrace{\begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & & \\ 0 & & 1 & 0 \end{pmatrix}}_{n}\Big\}n.$$

THE REASON FOR THIS FAILURE IS THAT $\mathrm{Tr}(A_\infty B_\infty)$ AND $\mathrm{Tr}(B_\infty A_\infty)$ ARE BOTH ∞! SO WE SHOULD EXPECT SOME TROUBLE BECAUSE OF A LACK OF CONVERGENCE.

EXAMPLES OF ∞-DIM VECTOR SPACES:

(1) $V = \ell_2(\mathbb{R})$ (SPACE OF SQUARE-SUMMABLE SEQUENCES IN $\mathbb{R}$)

DEF. $\underline{v} = (v_1, v_2, v_3, \dots) \in \ell_2(\mathbb{R})$ IF $\sum_{i=1}^{\infty} |v_i|^2 < \infty$, $v_i \in \mathbb{R}$.

- $\ell_2(\mathbb{R})$ IS A VECTOR SPACE SINCE

(i) $\underline{v}, \underline{w} \in \ell_2(\mathbb{R}) \Rightarrow \underset{(v_1+w_1, v_2+w_2, \dots)}{\underline{v}+\underline{w}} \in \ell_2(\mathbb{R})$

PF.

$$\sum_{i=1}^{\infty} |v_i + w_i|^2 = \sum_{i=1}^{\infty} |v_i|^2 + |w_i|^2 + \underbrace{2|v_i||w_i|}_{\leq |v_i|^2 + |w_i|^2} < \infty .$$

(SINCE FOR ANY $a, b \in \mathbb{R}$, $(|a| - |b|)^2 \geq 0$)

(ii) $\underline{v} \in \ell_2(\mathbb{R})$, c SCALAR $\Rightarrow c\underline{v} \in \ell_2(\mathbb{R})$.

PF. EASY.

- $\mathcal{E} = \{\underline{e}_i\}_{i=1}^{\infty}$, $\underline{e}_i = (0, \dots, 0, \overset{i\text{th coordinate}}{1}, 0, \dots)$ IS THE STANDARD BASIS OF $\ell_2(\mathbb{R})$. NOTE THAT ANY $\underline{v} \in \ell_2(\mathbb{R})$ CAN BE WRITTEN AS AN <u>INFINITE</u> LINEAR COMBINATION OF THE $\{\underline{e}_i\}$.

- $\ell_2(\mathbb{R})$ IS AN INNER PRODUCT SPACE WITH $\langle \underline{v} | \underline{w} \rangle = \sum_{i=1}^{\infty} v_i w_i$. NOTE THAT THIS IS A GENERALIZATION OF THE INNER PRODUCT OF $\mathbb{R}^n$ AS $n \to \infty$.

- SIMILARLY, ONE CAN DEFINE $\ell_2(\mathbb{C})$ AS ALL $\underline{v} = (v_1, v_2, v_3, \dots)$ S.T. $v_i \in \mathbb{C}$ AND $\sum_{i=1}^{\infty} |v_i|^2 < \infty$.

  THE INNER PRODUCT IS THEN $\langle \underline{v} | \underline{w} \rangle = \sum_{i=1}^{\infty} \overline{v_i}\, w_i$.

(2) $V = L_2(U)$ (SPACE OF SQUARE-INTEGRABLE FUNCTIONS $f: U \to \mathbb{R}$).

__DEF.__ $f \in L_2(U)$ IF $f: U \to \mathbb{R}$, AND $\int_U |f(x)|^2 dx < \infty$.

TYPICALLY, WE TAKE $U = [0, a]$ FOR SOME $a > 0$ OR $U = \mathbb{R}$.

- $L_2(U)$ IS A VECTOR SPACE, SINCE

  (i) $f, g \in L_2(U) \Rightarrow f+g \in L_2(U)$

  __PF.__

  $$\int_U |f(x)+g(x)|^2 dx \le \int_U \Big(|f(x)|^2 + |g(x)|^2 + \underbrace{2|f(x)||g(x)|}_{\le |f(x)|^2 + |g(x)|^2}\Big) dx < \infty.$$

  (ii) $cf \in L_2(U)$ FOR ANY SCALAR c.

  __PF.__ EASY.

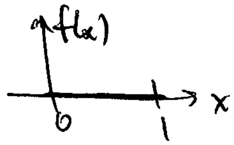
- WHAT IS A BASIS FOR $L_2(U)$? WE WILL SEE LATER WHEN DISCUSSING FOURIER SERIES.

- $L_2(U)$ is an inner product space with
$$\langle f | g \rangle = \int_U f(x)\, g(x)\, dx.$$

- Similarly, we can define $L_2(U; \mathbb{C})$ as all $f : U \to \mathbb{C}$ such that $\int_U |f(x)|^2\, dx < \infty$.

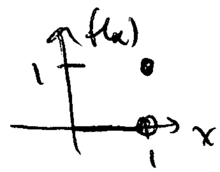
<u>Warning:</u> there are many subtleties when working with $L_2(U)$ which we will try to avoid. For ex., note that for $U = [0,1]$,

$$f = 0 \text{ for all } x \in [0,1] \Rightarrow \|f\|^2 = \langle f | f \rangle = \int_0^1 |f(x)|^2\, dx = 0,$$



as expected. However, we also see that

$$f(x) = \begin{cases} 0 & \text{for all } x \in [0,1) \\ 1 & \text{for } x = 1 \end{cases} \Rightarrow \|f\|^2 = \langle f | f \rangle = \int_0^1 |f(x)|^2\, dx = 0 \; !$$

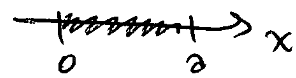


Therefore, positivity of the inner product does not hold as we have defined it since the integral does not "see" individual points on the line. To remedy this, we will think of the zero element as <u>any</u> f s.t. $\int_U |f|\, dx = 0$, and $f = g$ in $L_2(U)$ if $\int_U |f - g|\, dx = 0$.

Lecture 36
04/20/12

Q: What is a basis of $L^2(U)$ when

(a) $U = [0, a]$, $a > 0$?



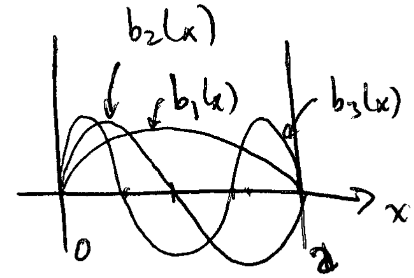
(b) $U = \mathbb{R}$?

As we will see, the answer in case (a) is significantly simpler than in case (b).

Fourier series on an interval (6.9):

Suppose $U = [0, a]$, $a > 0$.

Let $b_n(x) = \sin\left(\frac{n\pi x}{a}\right)$, $n = 1, 2, 3, \ldots$



- Easy to check that $b_n \in L_2([0, a])$ for each n.
- Furthermore, $\{b_n\}_{n=1}^{\infty}$ are orthogonal!

<u>Pf.</u> $\langle b_m | b_n \rangle = \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx$

(letting $y = \frac{x}{a}$) $= a \int_0^1 \sin(m\pi y) \sin(n\pi y)\, dy$

$= \frac{a}{2} \int_0^1 \left[\cos((n-m)\pi y) - \cos((n+m)\pi y)\right] dy$

(using the identity $2\sin(u)\sin(v) = \cos(u-v) - \cos(u+v)$)

$$= \begin{cases} \frac{a}{2}\left[\frac{\sin((n-m)\pi y)}{(n-m)\pi} - \frac{\sin((n+m)\pi y)}{(n+m)\pi}\right]_{y=0}^{y=1}, & m \neq n \\ \frac{a}{2}\left[1 - \frac{\sin(2n\pi y)}{2n\pi}\right]_0^1, & m = n \end{cases}$$

$$= \begin{cases} 0 & \text{IF } m \neq n \\ \frac{a}{2} & \text{IF } m = n. \end{cases}$$

$\Rightarrow \{b_n\}_{n=1}^{\infty}$ ORTHOGONAL (NOT ORTHONORMAL!). IN $L_2([0,a])$

- IN FACT, $\{b_n\}_{n=1}^{\infty}$ IS A BASIS OF $L_2([0,a])$.

"PF." LET $L = \frac{d^2}{dx^2}$. THEN AS WE HAVE SHOWN PREVIOUSLY, L IS SELF-ADJOINT ON $L_2([0,a])$.

NOTE THAT $\{b_n\}_{n=1}^{\infty}$ ARE "EIGENVECTORS" (WE WILL CALL THEM EIGENFUNCTIONS) OF L W/ CORRESPONDING EIGENVALUES $\lambda_n = -\frac{n^2\pi^2}{a^2}$ SINCE $L\, b_n(x) = \lambda_n b_n(x)$.

ASSUMING THE SPECTRAL THM. HOLDS FOR SELF-ADJOINT OPERATORS ON ∞-DIM. SPACES, THIS IMPLIES THAT $\{b_n\}_{n=1}^{\infty}$ IS AN ORTHOGONAL BASIS OF $L_2([0,a])$.

TECHNICALLY, L IS SELF-ADJOINT ON THE SPACE

$$C_0^\infty([0,a]) = \{ f : [0,a] \to \mathbb{R} : f(0) = f(a) = 0 \text{ AND } f \text{ INFINITELY DIFFERENTIABLE} \}$$

THIS, WITH THE FACT THAT FOR ANY $f \in L_2([0,a])$ THERE IS A SEQUENCE $\{g_m\}_{m=1}^\infty \subset C_0^\infty([0,a])$ THAT APPROXIMATES f (I.E., $\|f - g_m\|_{L_2([0,a])} = \int_0^a |f(x) - g_m(x)|^2 dx \xrightarrow[m\to\infty]{} 0$.)

IMPLIES THAT $\{b_n\}_{n=1}^\infty$ IS A BASIS OF $L_2([0,a])$.

- NOTE THAT WE CAN NOW WRITE

$$f(x) = \sum_{n=1}^\infty c_n b_n(x) = \sum_{n=1}^\infty c_n \sin\left(\frac{n\pi x}{a}\right)$$

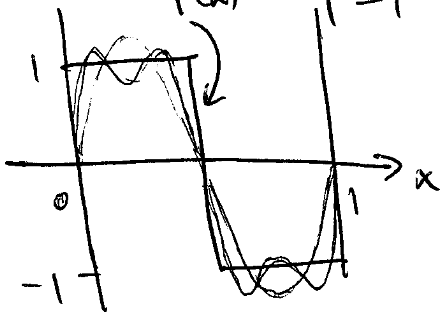
FOR ANY $f \in L_2([0,a])$. THE c_n'S ARE CALLED FOURIER COEFFICIENTS OF f AND THE ABOVE REPRESENTATION IS CALLED FOURIER SERIES.

- TO FIND c_n, WE NOTE THAT

$$c_n = \frac{\langle b_n | f \rangle}{\langle b_n | b_n \rangle} = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx .$$

Ex.

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}) \\ -1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases} .$$



$$\Rightarrow \quad c_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$$

$$= 2\left(\int_0^{\frac{1}{2}} \sin(n\pi x)\, dx - \int_{\frac{1}{2}}^{1} \sin(n\pi x)\, dx \right)$$

$$= \frac{2}{n\pi}\left(\Big[-\cos(n\pi x)\Big]_0^{\frac{1}{2}} + \Big[\cos(n\pi x)\Big]_{\frac{1}{2}}^{1} \right)$$

$$= \frac{2}{n\pi}\left(1 - 2\cos\left(\frac{n\pi}{2}\right) + \cos(n\pi)\right),$$

$$n = 1, 2, 3, \ldots$$

NOTE THAT $|c_n| \sim \frac{1}{n}$ AS $n \to \infty$.

WITH ONLY A FEW TERMS OF THE SERIES WE CAN GET A FAIRLY GOOD APPROXIMATION TO OUR ORIGINAL FUNCTION — I.E.

$$f(x) \approx \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{a}\right) .$$

LECTURE 37
04/23/12

RECALL THE FOURIER BASIS $\{b_n\}_{n=1}^{\infty}$ OF $L_2([0,a])$ GIVEN BY $b_n(x) = \sin\left(\frac{n\pi x}{a}\right)$, $n = 1, 2, 3, \ldots$. ANY FUNCTION $f \in L_2([0,a])$ CAN BE WRITTEN AS $f(x) = \sum_{n=1}^{\infty} c_n b_n(x)$, WHERE $c_n = \frac{\langle b_n | f \rangle}{\langle b_n | b_n \rangle} = \frac{2}{a}\int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$.

NOTE: THE FOURIER COEFFICIENTS c_n TELL US "HOW MUCH" OF THE TERM $\sin\left(\frac{n\pi x}{a}\right)$ IS PRESENT IN $f(x)$ — IN PARTICULAR, NOTE THAT $c_n \to 0$ AS $n \to \infty$ (OTHERWISE THE SERIES COULD NOT CONVERGE!) IN ADDITION, THE RATE OF DECAY OF THE c_n'S AS $n \to \infty$ DEPENDS ON HOW SMOOTH f IS SINCE LARGE DERIVATIVES CAN ONLY BE CAPTURED BY HIGH FREQUENCY (I.E, LARGE n) TERMS.

$|f'|$ LARGE $\Rightarrow$ HIGH FREQUENCIES IMPORTANT

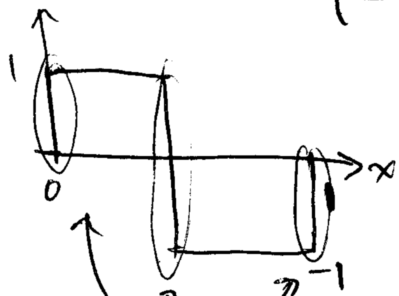
$\Rightarrow c_n \to 0$ (AS $n \to \infty$) SLOWLY

$|f'|$ SMALL $\Rightarrow$ HIGH FREQUENCIES UNIMPORTANT

$\Rightarrow c_n \to 0$ (AS $n \to \infty$) QUICKLY.

<u>Ex.</u>

$$f(x) = \begin{cases} 1 & , \quad x \in [0, \tfrac{1}{2}) \\ -1 & , \quad x \in [\tfrac{1}{2}, 1] \end{cases}$$



$|f'| \gg 1.$

WE SAW LAST TIME THAT

$$c_n = \frac{2}{n\pi}\left(1 - 2\cos\left(\frac{n\pi}{2}\right) + \cos(n\pi)\right)$$

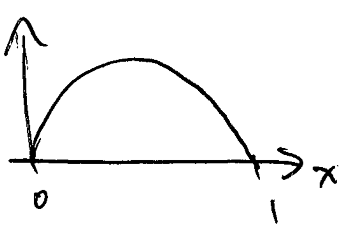
$$\sim \frac{1}{n} \quad \text{AS} \quad n \to \infty.$$

" PROPORTIONAL TO "

$\Rightarrow$ SLOW DECAY DUE TO LARGE DERIVATIVES $|f'|$ AT $x = 0, \frac{1}{2}, 1$.

<u>Ex.</u>

$$f(x) = x - x^2, \quad x \in [0, 1].$$



CALCULATING THE COEFFICIENTS WE GET

$$c_n = 2\int_0^1 (x - x^2)\sin(n\pi x)\,dx$$

$$= \frac{4(1 - \cos(n\pi))}{n^3\pi^3}$$

$$\sim \frac{1}{n^3} \quad \text{AS} \quad n \to \infty$$

$\Rightarrow$ FAST DECAY SINCE $|f'|$ IS EVERYWHERE BOUNDED BY 2.

PROPERTIES :

(i) BASIS $\mathcal{B} = \{b_n\}_{n=1}^{\infty}$ MAKES $L_2([0,a])$ LOOK LIKE l_2 !

IF $f, g \in L_2([0,a])$, $f(x) = \sum_{n=1}^{\infty} c_n b_n(x)$,
$g(x) = \sum_{n=1}^{\infty} d_n b_n(x)$

$$\langle f | g \rangle_{L_2} = \langle \sum_{n=1}^{\infty} c_n b_n \Big| \sum_{m=1}^{\infty} d_m b_m \rangle_{L_2}$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_n d_m \langle b_n | b_m \rangle_{L_2}$$
$$= \frac{a}{2} \sum_{n=1}^{\infty} c_n d_n = \frac{a}{2} \langle [f]_{\mathcal{B}} | [g]_{\mathcal{B}} \rangle_{l_2}$$

SINCE $[f]_{\mathcal{B}} = (c_1, c_2, \dots) \in l_2$.
$[g]_{\mathcal{B}} = (d_1, d_2, \dots) \in l_2$.

$$\Rightarrow \boxed{\|f\|^2_{L_2([0,a])} = \frac{a}{2} \| [f]_{\mathcal{B}} \|^2_{l_2}}$$

(PARSEVAL'S THM.).

IN PARTICULAR, THIS QUANTIFIES THE ERROR BETWEEN A FUNCTION AND ITS FOURIER SERIES (TRUNCATED AT N^{th} TERM) AS

$$\text{ERROR}^2 = \|f - f_N\|^2_{L_2} = \Big\| \sum_{n=N+1}^{\infty} c_n b_n \Big\|^2_{L_2} = \frac{a}{2} \sum_{n=N+1}^{\infty} |c_n|^2 .$$

(ii) FOURIER SERIES $f = \sum c_n b_n$ IS IDEAL WHEN WORKING WITH EVEN DERIVATIVES!

INDEED, RECALL THAT b_n IS THE nTH EIGENFUNCTION OF $L = \frac{d^2}{dx^2}$ w/ CORRESPONDING EIGENVALUE $\lambda_n = \frac{-n^2\pi^2}{a^2}$.

THEN,

$$
\begin{aligned}
f''(x) = Lf(x) &= L\left(\sum_{n=1}^{\infty} c_n b_n(x)\right) \\
&= \sum_{n=1}^{\infty} c_n L b_n(x) \\
&= \sum_{n=1}^{\infty} c_n \lambda_n b_n(x).
\end{aligned}
$$

SO, THE nTH FOURIER COEFFICIENT OF f'' IS SIMPLY $\lambda_n c_n = -\frac{n^2\pi^2}{a^2} c_n$ (I.E., DERIVATIVES LEAD TO A SIMPLE MULTIPLICATION IN TERMS OF FOURIER COEFFICIENTS!

Ex. CONSIDER THE PARTIAL DIFFERENTIAL EQUATION (PDE)

$$
\begin{cases}
\partial_t u = \partial_{xx} u \rightsquigarrow \text{"HEAT EQN."} \\
u(0,t) = u(a,t) = 0 \quad \text{(BOUNDARY CONDITIONS)} \\
u(x,0) \text{ GIVEN.} \quad \text{(INITIAL CONDITIONS)}
\end{cases}
$$

HERE, $u(x,t)$ IS TO BE THOUGHT OF AS THE TEMPERATURE OF A THIN WIRE AT LOCATION $x \in [0,a]$ AT TIME $t \geq 0$.

THE BOUNDARY CONDITIONS $u(0,t) = u(a,t) = 0$ IMPLY THAT AT ALL TIMES THERE IS NO HEAT AT THE BOUNDARIES OF THE WIRE. SINCE HEAT SPREADS, WE SHOULD EXPECT THE SOLUTION $u(x,t) \to 0$ FOR ALL $x \in [0,a]$ AS $t \to \infty$ SINCE HEAT IS SUCKED OUT OF THE WIRE AT THE BOUNDARIES.

WE FIND THE SOLUTION BY SOLVING FOR THE EVOLUTION OF FOURIER COEFF.:

ASSUME $$u(x,t) = \sum_{n=1}^{\infty} c_n(t)\, b_n(x)$$ (SEPARATION OF VARIABLES)

$c_n(t)$: TIME-DEPENDENT FOURIER COEFF.

$b_n(x)$: FOURIER BASIS

$$\Rightarrow \quad \partial_t u(x,t) = \sum_{n=1}^{\infty} \left(\frac{d}{dt} c_n(t)\right) b_n(x)$$

$$\partial_{xx} u(x,t) = \sum_{n=1}^{\infty} c_n(t)\, L\, b_n(x) \qquad \left(L = \frac{d^2}{dx^2}\right)$$

$$= \sum_{n=1}^{\infty} (\lambda_n c_n(t))\, b_n(x)$$

$$\Rightarrow \quad \frac{d}{dt} c_n(t) = \lambda_n c_n(t) \qquad \text{FOR ALL } n = 1,2,3,\ldots$$

$$\Rightarrow \quad c_n(t) = c_n(0)\, e^{\lambda_n t} = c_n(0)\, e^{-\left(\frac{n^2\pi^2}{a^2}\right)t}, \quad n = 1,2,3,\ldots$$

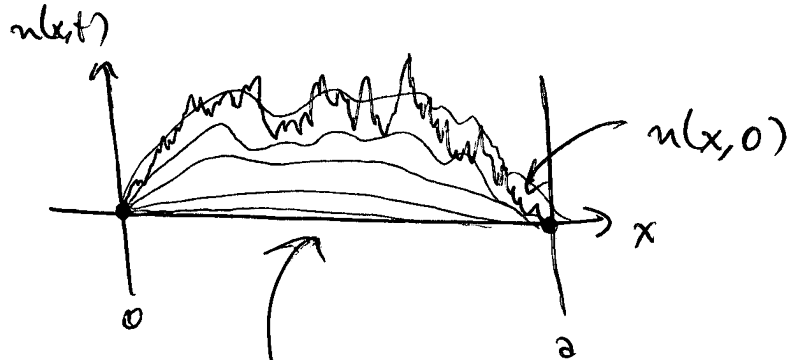
WHERE $u(x,0) = \sum_{n=1}^{\infty} c_n(0)\, b_n(x)$ <u>GIVEN</u>.

$$c_n(0) = \frac{2}{a}\int_0^a u(x,0)\, b_n(x)\, dx.$$

NOTE THAT $c_n(t) \to 0$ AS $t \to \infty$ FOR EVERY n,

BUT THAT THE DECAY IS SIGNIFICANTLY FASTER FOR LARGER FREQUENCIES n!

$\Rightarrow$ SOLUTION GETS SMOOTHER AS $t \to \infty$ AND GOES TO 0 EVERYWHERE.



SOLUTION GETS SMOOTHER IN x AS TIME PROGRESSES, GOES TO 0 AS $t \to \infty$.

NOTE: THIS IS THE ∞-DIM ANALOGUE OF CONTINUOUS-TIME EVOLUTION IN $\mathbb{R}^n$!

$$\begin{cases} \frac{d\underline{u}}{dt} = A\underline{u} \\ u(0) \text{ given} \end{cases}, \quad A : \mathbb{R}^n \to \mathbb{R}^n.$$

WRITING $\underline{u}(t) = \sum_{i=1}^{n} c_i(t)\, \underline{b}_i$ ← i^{th} E-VECTOR OF A

WE FOUND $c_i(t) = c_i(0)\, e^{\lambda_i t}$. ← i^{th} E-VALUE OF A

- STABILITY OF i^{th} MODE $\underline{b}_i$ WAS DETERMINED BY $\mathrm{Re}(\lambda_i)$.

LECTURE 38
04/27/12

Q: WE HAVE FOUND ONE BASIS OF $L_2([0,a])$.
ARE THERE OTHERS, AND HOW TO FIND THEM?

FOURIER SERIES (CONT'D) (8.5, 8.7)

LET US CAREFULLY REDERIVE THE BASIS WE HAVE WORKED WITH SO FAR:

① FOURIER SINE BASIS: DEFINE THE SPACE

$$L_2^{DIR}([0,a]) = \{ f \in L_2([0,a]) : \overbrace{f(0)=0,\ f(a)=0}^{\text{"DIRICHLET BOUNDARY CONDITIONS"}},\ f \text{ INFINITELY DIFFERENTIABLE.} \}$$

NOTE THAT $L_2^{DIR}([0,a]) \subseteq L_2([0,a])$.

- IN FACT, THIS SUBSPACE "WELL-APPROXIMATES" $L_2([0,a])$ IN THAT FOR ANY $f \in L_2([0,a])$, THERE IS A SEQUENCE $\{f_n\}_{n=1}^{\infty} \subseteq L_2^{DIR}([0,a])$ SUCH THAT $\|f - f_n\|_{L_2}^2 \xrightarrow{n\to\infty} 0$ (I.E, WE CAN APPROX. f WITHIN AN ARBITRARILY SMALL ERROR BY AN ELEMENT OF $L_2^{DIR}([0,a])$.

- $L = \frac{d^2}{dx^2}$ IS SELF-ADJOINT ON $L_2^{DIR}([0,a])$.

PF. $\langle f | L g \rangle = \int_0^a f(x)\, g''(x)\, dx$

$f, g \in L_2^{DIR}([0,a])$

$$= \underbrace{\cancel{\left[f(x)\, g'(x)\right]_0^a}}_{\substack{=0 \text{ SINCE} \\ f(0)=0,\ f(a)=0}} - \int_0^a f'(x)\, g'(x)\, dx$$

$$= \underbrace{\cancel{\left[-f'(x)\, g(x)\right]_0^a}}_{\substack{=0 \text{ SINCE} \\ g(0)=0,\ g(a)=0}} + \int_0^a f''(x)\, g(x)\, dx$$

$$= \langle Lf | g \rangle. \quad \checkmark$$

- WHAT ARE EIGENFUNCTIONS OF L ON $L_2^{DIR}([0,a])$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{DIR}([0,a]) \end{cases} \implies \begin{cases} f''(x) = \lambda f(x) \\ f(0)=0,\ f(a)=0 \end{cases}$$

$$\implies f(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, 3, \ldots$$

$$\text{w/ } \lambda = -\frac{n^2\pi^2}{a^2}.$$

THIS GIVES THE ~~ORTH~~ ORTHOGONAL BASIS (BY SELF-ADJOINTNESS OF L) $\{b_n\}_{n=1}^{\infty}$ ON $L_2^{DIR}([0,a])$ (AND THUS ON $L_2([0,a])$)

$$b_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad n=1,2,3,\ldots$$

w/ EIGENVALUES $\lambda_n = \dfrac{-n^2\pi^2}{a^2}$.

(FOURIER SINE BASIS)

$$\rightsquigarrow f(x) = \sum_{n=1}^{\infty} c_n b_n(x), \quad c_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx.$$

② STANDARD FOURIER BASIS (REAL): DEFINE THE SPACE

$$L_2^{PER}([0,a]) = \{f \in L_2([0,a]) : \underbrace{f(0)=f(a),\ f'(0)=f'(a)}_{\text{"PERIODIC BOUNDARY CONDITIONS"}},\ f \text{ INFINITELY DIFFERENTIABLE}\}$$

NOTE THAT $L_2^{PER}([0,a]) \subseteq L_2([0,a])$.

- IN FACT, AS WITH $L_2^{DIR}([0,a])$, WE HAVE THAT $L_2^{PER}([0,a])$ WELL-APPROXIMATES $L_2([0,a])$.
- $L = \dfrac{d^2}{dx^2}$ IS SELF-ADJOINT ON $L_2^{PER}([0,a])$.

PF. $\langle f | Lg \rangle = \int_0^a f(x)\, g''(x)\, dx$

$f, g \in L_2^{PER}([0,a])$.

$$= \underbrace{\cancel{\left[f(x)\, g'(x)\right]_0^a}}_{\substack{=0 \text{ SINCE } f(0)=f(a), \\ g'(0)=g'(a)\,!}} - \int_0^a f'(x)\, g'(x)\, dx$$

$$= \underbrace{\cancel{\left[-f(x)\, g'(x)\right]_0^a}}_{\substack{=0 \text{ SINCE } f'(0)=f'(a) \\ g(0)=g(a)\,!}} + \int_0^a f''(x)\, g(x)\, dx$$

$$= \langle Lf | g \rangle .$$

- WHAT ARE EIGENFUNCTIONS OF L ON $L_2^{PER}([0,a])$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{PER}([0,a]) \end{cases} \Rightarrow \begin{cases} f''(x) = \lambda f(x) \\ f(0) = f(a),\ f'(0) = f(a). \end{cases}$$

$$\Rightarrow f(x) = 1 \quad w/\ \lambda = 0 \quad \underline{\text{OR}}$$

$$f(x) = \sin\left(\frac{2n\pi x}{a}\right) \quad w/\ \lambda = -\frac{4n^2\pi^2}{a^2} \quad \underline{\text{OR}}$$

$$f(x) = \cos\left(\frac{2n\pi x}{a}\right) \quad w/\ \lambda = -\frac{4n^2\pi^2}{a^2}.$$

THIS GIVES THE ORTHOGONAL BASIS FOR $L_2^{PER}([0,a])$ (AND THUS FOR $L_2([0,a])$)

$$\left\{ 1, \left\{ \sin\left(\frac{2n\pi x}{a}\right) \right\}_{n=1}^{\infty}, \left\{ \cos\left(\frac{2n\pi x}{a}\right) \right\}_{n=1}^{\infty} \right\}$$

W/ E-VALUES

$$\lambda_0 = 0, \quad \lambda_n = \frac{-4n^2\pi^2}{a^2} \; ! \quad n = 1, 2, 3, \ldots$$

(STANDARD FOURIER BASIS)

THAT IS, EIGENVALUE λ_n HAS GEOMETRIC MULTIPLICITY 2 (FOR $n = 1, 2, 3, \ldots$), WITH CORRESPONDING EIGENSPACE

$$E_{\lambda_n} = \text{SPAN} \left\{ \sin\left(\frac{2n\pi x}{a}\right), \cos\left(\frac{2n\pi x}{a}\right) \right\}.$$

$$\rightsquigarrow f(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left[\alpha_n \cos\left(\frac{2n\pi x}{a}\right) + \beta_n \sin\left(\frac{2n\pi x}{a}\right) \right],$$

"ALPHA" $\curvearrowright$ $$\alpha_n = \frac{2}{a} \int_0^a f(x) \cos\left(\frac{2n\pi x}{a}\right) dx,$$

"BETA" $\curvearrowright$ $$\beta_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{2n\pi x}{a}\right) dx.$$

(3) STANDARD FOURIER BASIS (COMPLEX):

CONSIDER THE SPACE OF COMPLEX-VALUED SQUARE-INTEGRABLE FUNCTIONS $L_2([0,a]; \mathbb{C})$ W/ INNER PRODUCT

$$\langle f | g \rangle = \int_0^a \overline{f(x)}\, g(x)\, dx.$$

DEFINE THE SPACE

$$L_2^{PER}([0,a];\mathbb{C}) = \{ f \in L_2([0,a];\mathbb{C}) : f(0)=f(a),\ f \text{ INFINITELY DIFF.} \}$$

AS BEFORE, $L_2^{PER}([0,a];\mathbb{C}) \subseteq L_2([0,a];\mathbb{C})$ WELL-APPROXIMATES. THEN, WE FIND THAT:

- $L = -i\dfrac{d}{dx}$ IS SELF-ADJOINT ON $L_2^{PER}([0,a];\mathbb{C})$.

PF.

$f,g \in L_2^{PER}([0,a];\mathbb{C})$

$$\langle f | Lg \rangle = -i\int_0^a \overline{f(x)}\, g'(x)\,dx$$

$$= \underbrace{\cancel{\left[-i\overline{f(x)}\,g(x)\right]_0^a}}_{=0 \text{ SINCE } \overline{f(0)}=\overline{f(a)},\ g(0)=g(a)} + i\int_0^a \overline{f'(x)}\, g(x)\,dx$$

$$= \int_0^a \overline{(-if'(x))}\, g(x)\,dx$$

$$= \langle Lf | g \rangle .$$

✓

- WHAT ARE THE EIGENFUNCTIONS OF L ON $L_2^{PER}([0,a];\mathbb{C})$?

$$\begin{cases} Lf = \lambda f \\ f \in L_2^{PER}([0,a];\mathbb{C}) \end{cases} \Rightarrow \begin{cases} -if'(x) = \lambda f(x) \\ f(0) = f(a) \end{cases}$$

$$\Rightarrow \quad f(x) = \exp\left(\frac{2\pi i n x}{a}\right), \quad n \in \{\dots, -1, 0, 1, \dots\} = \mathbb{Z}.$$

$$w/ \ \lambda_n = \frac{2\pi n}{a}.$$

THIS GIVES THE ORTHOGONAL BASIS FOR $L_2^{PER}([0,a]; \mathbb{C})$ (AND THUS FOR $L_2([0,a]; \mathbb{C})$)

$$\boxed{\left\{ \exp\left(\frac{2\pi i n x}{a}\right) \right\}_{n=-\infty}^{n=\infty} \quad w/ \text{ E-VALUES } \lambda_n = \frac{2\pi n}{a}}$$

(STD. FOURIER BASIS (COMPLEX))

$$\rightsquigarrow \quad f(x) = \sum_{n=-\infty}^{\infty} \gamma_n \exp\left(\frac{2\pi i n x}{a}\right),$$

"GAMMA" $\rightarrow$ $$\gamma_n = \frac{1}{a} \int_0^a f(x) \exp\left(\frac{-2\pi i n x}{a}\right) dx.$$

NOTE: THE STD. FOURIER BASIS IN THE REAL CASE IS A SPECIAL CASE OF WHAT WE NOW HAVE.

USING THAT

$$\exp\left(\frac{2\pi i n x}{a}\right) = \cos\left(\frac{2n\pi x}{a}\right) + i \sin\left(\frac{2n\pi x}{a}\right),$$

WE HAVE THAT FOR $n = 0, 1, 2, \dots$

$$\alpha_n = \gamma_n + \gamma_{-n}, \quad \beta_n = +i(\gamma_n - \gamma_{-n}).$$