M346 (92153), Sample Final Exam Solutions

1.

a) Consider $\mathbb{R}^3$ with the standard inner product. Convert the basis $\mathcal{B} = \{(1,2,0)^T, (3,1,1)^T, (4,3,-5)^T\}$ into an orthonormal basis.

Solution: Using Gram-Schmidt, an orthonormal basis is $\mathcal{E} = \left\{\frac{1}{\sqrt{5}}(1,2,0)^T, \frac{1}{\sqrt{6}}(2,-1,1)^T, \frac{1}{\sqrt{30}}(2,\,-1,\,-5)^T\right\}$. Your answer may be different if the order of vectors in your orthogonalization procedure is different from the obvious one.

b) Find the matrix of the projection P_W onto the subspace $W = \operatorname{span}\{(1,\,2,\,0)^T, (3,\,1,\,1)^T\}$. Use this to compute $P_{W^\perp}\boldsymbol{v}$, where $\boldsymbol{v} = (1,2,3)^T$, where $W^\perp$ is the orthogonal complement of W (the subspace of all vectors orthogonal to W).

Solution: $P_W = P_{\boldsymbol{e}_1} + P_{\boldsymbol{e}_2} = |\boldsymbol{e}_1\rangle\langle\boldsymbol{e}_1| + |\boldsymbol{e}_2\rangle\langle\boldsymbol{e}_2| = \frac{1}{5}\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}$.

c) On $\mathbb{R}_2[t]$ with inner product $\langle p|q\rangle = \int_0^2 p(t)q(t)dt$, transform $\{1,\,t,\,t^2\}$ into an orthogonal basis (does not need to be orthonormal).

Solution: $\mathcal{D} = \{1, t-1, t^2 - 2t + 2/3\}$.

2.

a) Find the equation of the best line through the points $(1,-4)$, $(2,1)$, and $(3,2)$. Is this line unique?

Solution: Fitting the model $y = c + dx$ we have that $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $\boldsymbol{b} = (-4,1,2)^T$, so $A^*A = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$ and $A^*\boldsymbol{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$. Solving the normal equation $A^*A\boldsymbol{x}_{\text{LS}} = A^*\boldsymbol{b}$ gives the unique least-squares solution $\boldsymbol{x}_{\text{LS}} = (-19/3, 3)^T$ so the best line is $y = -19/3 + 3x$.

b) Let W be the subspace of $\mathbb{R}^3$ spanned by $(1,\,2,\,3)^T$ and $(1,\,1,\,1)^T$. Find the point in W which lies closest to $(-4,1,2)^T$. Justify your answer.

Solution: The closest point to $\boldsymbol{b}$ which lies in $\operatorname{Ran}(A)$ is $A\boldsymbol{x}_{\text{LS}} = (-10/3, -1/3, 8/3)^T$.

3. Let $A = \begin{pmatrix} 4 & 2 & -2 & 2 \\ 3 & -1 & 2 & -3 \end{pmatrix}$.

a) What is the rank r of A?

Solution: $r = 2$.

b) Write the singular value decomposition (SVD) of A as a sum of r terms (you do not need to expand your answers as a matrix). [Hint: Remember that the eigenvalues and eigenvectors of A^*A and AA^* are intimately related! Choose the easiest matrix to work with.]

Solution: We work with AA^* since this is a smaller matrix than A^*A. The eigenvalues of AA^* are $\sigma_1 = 2\sqrt{7}$ and $\sigma_2 = \sqrt{23}$, with corresponding orthonormal eigenvectors $\boldsymbol{u}_1 = (1, 0)^T$ and $\boldsymbol{u}_2 = (0,1)^T$. Then A^*A has the same eigenvalues with corresponding eigenvectors $\boldsymbol{v}_1 = \frac{1}{\sigma_1}A^*\boldsymbol{u}_1 = \frac{1}{\sqrt{7}}(2, 1, -1, 1)^T$ and $\boldsymbol{v}_2 = \frac{1}{\sigma_2}A^*\boldsymbol{u}_2 = \frac{1}{\sqrt{23}}(3, -1, 2, 3)^T$. So the SVD of A is $A = \sum_{i=1}^{2} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^*$.

c) Compute the error between A and its best rank-one approximation.

Solution: Since the best rank-one approximation is $A_1 = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^*$, the approximation error is $\|A - A_1\| = \sqrt{\sigma_2^2} = \sqrt{23}$ in the Frobenius norm.

4. Consider the symmetric matrix $A = \begin{pmatrix} 24 & 7 \\ 7 & -24 \end{pmatrix}$.

a) Write $A = UDU^*$ for an appropriate diagonal matrix D and unitary matrix U.

Solution: $D = \begin{pmatrix} 25 & 0 \\ 0 & -25 \end{pmatrix}$, $U = \frac{1}{5\sqrt{2}}\begin{pmatrix} 7 & 1 \\ 1 & -7 \end{pmatrix}$.

b) Express $\boldsymbol{x} = (13, 9)^T$ as a linear combination of the eigenvectors found in part (a).

Solution: $\boldsymbol{x} = 5\sqrt{2}(2\boldsymbol{u}_1 - \boldsymbol{u}_2)$ where $\boldsymbol{u}_1, \boldsymbol{u}_2$ are the columns of U.

c) Let $|A| = U|D|U^*$, where $|D|$ is the diagonal matrix of *magnitudes* of the eigenvalues of A. Show that $|A|$ is positive and compute $\sqrt{|A|}$.

Solution: $|A| = U|D|U^*$ with $|D| = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$. It is easy to see that $|A|$ is self adjoint and has nonnegative eigenvalues, and is therefore positive. Then we have that $\sqrt{|A|} = U|D|^{1/2}U^* = \frac{1}{50}\begin{pmatrix} 7 & 1 \\ 1 & -7 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 1 & -7 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$.

5. True or false? Justify your answers.

a) The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ has orthogonal eigenvectors.

Solution: True. This holds by the spectral theorem since the matrix is normal.

b) $\frac{1}{\sqrt{7}}\begin{pmatrix} 2-i & -1+i \\ 1+i & 2+i \end{pmatrix}$ is unitary.

Solution: True. The columns of the matrix are orthonormal.

c) If a matrix $A \in M_{n,n}(\mathbb{C})$ satisfies $A = A^T$ then the eigenvalues of A are necessarily real.

Solution: False. If the entries are complex then this does not necessarily hold.

d) If $\langle f|g\rangle = \int_0^\infty f(x)g(x)e^{-x}dx$ for functions f, $g \in L_2([0,\infty))$ and $L = x + \frac{d}{dx}$ (assume that all elements of $L_2([0,\infty))$ are differentiable), its adjoint is $L^* = x - \frac{d}{dx}$.

Solution: False. Integration by parts shows that the adjoint is actually $L^* = (x+1) - \frac{d}{dx}$.

6.

i. For which $z \in \mathbb{R}$ is the sequence $\boldsymbol{v} = (a_1, a_2, a_3, \ldots)$, $a_n = z^n$, in $l_2(\mathbb{R})$? Why?

Solution: Since $\|\boldsymbol{v}\|_{l_2(\mathbb{R})}^2 = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |z|^2$, the series converges if and only if $|z| < 1$ (geometric series).

ii. For which $p \geq 0$ is the sequence $\boldsymbol{v} = (a_1, a_2, a_3, \ldots)$, $a_n = (2 + n^p)^{-1}$, in $l_2(\mathbb{R})$? Why?

Solution: Since $\|\boldsymbol{v}\|_{l_2(\mathbb{R})}^2 = \sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} \left|\frac{1}{2 + n^p}\right|^2$, the series converges if and only if $p > 1/2$ by the limit comparison test for infinite series.

7. Compute the Fourier sine series of the function $f(x) = \cos(\pi x)$ on the interval [0, 1]. [Hint: Use the trigonometric identity $2\sin(u)\cos(v) = \sin(u+v) + \sin(u-v)$, if needed.]

Solution: $\cos(\pi x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$, where $c_n = \frac{2n}{\pi}\left[\frac{1 + (-1)^n}{n^2 - 1}\right]$.

8. Using Fourier sine series, find the solution $u(x, t)$ to the time-dependent Schrodinger equation for a free particle in a 1-dimensional box:

$$\begin{cases} \partial_t u = i\partial_{xx} u \\ u(0, t) = 0, u(a, t) = 0 \ , \qquad x \in [0, a],\ t \geq 0. \\ u(x, 0) \text{ given} \end{cases}$$

(Here, $i = \sqrt{-1}$ is the imaginary constant.) That is, find the Fourier coefficients of the solution in terms of the Fourier coefficients of the initial data $u(x,\ 0)$. Are the modes of the system stable, neutrally stable, or unstable? How does the solution behave and how does this differ from the heat equation studied earlier?

Solution: The solution is $u(x,\ t) = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi x}{a}\right)$ with $c_n(t) = e^{i\lambda_n t} c_n(0)$, where $\lambda_n = -\frac{n^2\pi^2}{a^2}$ and $\{c_n(0)\}_{n=1}^{\infty}$ are the Fourier coefficients of the initial data $u(x,\ 0)$. We therefore see that the modes $\left\{\sin\left(\frac{n\pi x}{a}\right)\right\}_{n=1}^{\infty}$ of the system are all neutrally stable since $\mathrm{Re}(i\lambda_n) = 0$ for all n. Using Euler's formula, we see that the solution takes the form

$$u(x, t) = \sum_{n=1}^{\infty} \left\{ a_n \sin\left(\frac{n^2\pi^2 t}{a^2}\right) \sin\left(\frac{n\pi x}{a}\right) + b_n \cos\left(\frac{n^2\pi^2 t}{a^2}\right) \sin\left(\frac{n\pi x}{a}\right) \right\}$$

for some set of complex-valued constants $\{a_n,\ b_n\}_{n=1}^{\infty}$ which describes a wave in space and time (called a plane wave). This is significantly different from the behavior of the heat equation, where all modes of the system decayed and the solution converges to 0 everywhere as $t \to \infty$.