M346 (92153), Sample Final Exam Questions

Below are some sample final exam questions that cover only the material taught since the last midterm (for material taught prior to this, rework your midterms and sample midterms). Please note that the intent of these is to help prepare for the exam, and that actual exam questions will not merely be modifications of these problems. Consult your HW, lecture notes, and books for additional sources of material to review.

1.

a) Consider $\mathbb{R}^3$ with the standard inner product. Convert the basis $\mathcal{B} = \{(1,2,0)^T, (3,1,1)^T, (4,3,-5)^T\}$ into an orthonormal basis.

b) Find the matrix of the projection P_W onto the subspace $W = \text{span}\{(1,2,0)^T, (3,1,1)^T\}$. Use this to compute $P_{W^\perp}\boldsymbol{v}$, where $\boldsymbol{v} = (1,2,3)^T$, where $W^\perp$ is the orthogonal complement of W (the subspace of all vectors orthogonal to W).

c) On $\mathbb{R}_2[t]$ with inner product $\langle p|q\rangle = \int_0^2 p(t)q(t)dt$, transform $\{1, t, t^2\}$ into an orthogonal basis (does not need to be orthonormal).

2.

a) Find the equation of the best line through the points $(1,-4)$, $(2,1)$, and $(3,2)$. Is this line unique?

b) Let W be the subspace of $\mathbb{R}^3$ spanned by $(1, 2, 3)^T$ and $(1, 1, 1)^T$. Find the point in W which lies closest to $(-4,1,2)^T$. Justify your answer.

3. Let $A = \begin{pmatrix} 4 & 2 & -2 & 2 \\ 3 & -1 & 2 & -3 \end{pmatrix}$.

a) What is the rank r of A?

b) Write the singular value decomposition (SVD) of A as a sum of r terms (you do not need to expand your answers as a matrix). [Hint: Remember that the eigenvalues and eigenvectors of A^*A and AA^* are intimately related! Choose the easiest matrix to work with.]

c) Compute the error between A and its best rank-one approximation.

4. Consider the symmetric matrix $A = \begin{pmatrix} 24 & 7 \\ 7 & -24 \end{pmatrix}$.

a) Write $A = UDU^*$ for an appropriate diagonal matrix D and unitary matrix U.

b) Express $\boldsymbol{x} = (13, 9)^T$ as a linear combination of the eigenvectors found in part (a).

c) Let $|A| = U|D|U^*$, where $|D|$ is the diagonal matrix of *magnitudes* of the eigenvalues of A. Show that $|A|$ is positive and compute $\sqrt{|A|}$.

5. True or false? Justify your answers.

a) The matrix $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ has orthogonal eigenvectors.

b) $\frac{1}{\sqrt{7}}\begin{pmatrix} 2-i & -1+i \\ 1+i & 2+i \end{pmatrix}$ is unitary.

c) If a matrix $A \in M_{n,n}(\mathbb{C})$ satisfies $A = A^T$ then the eigenvalues of A are necessarily real.

d) If $\langle f | g \rangle = \int_0^\infty f(x)g(x)e^{-x}dx$ for functions $f,\, g \in L_2([0,\, \infty))$ and $L = x + \frac{d}{dx}$ (assume that all elements of $L_2([0, \infty))$ are differentiable), its adjoint is $L^* = x - \frac{d}{dx}$.

6.

i. For which $z \in \mathbb{R}$ is the sequence $\boldsymbol{v} = (a_1, a_2, a_3, \ldots)$, $a_n = z^n$, in $l_2(\mathbb{R})$? Why?

ii. For which $p \geq 0$ is the sequence $\boldsymbol{v} = (a_1, a_2, a_3, \ldots)$, $a_n = (2 + n^p)^{-1}$, in $l_2(\mathbb{R})$? Why?

7. Compute the Fourier sine series of the function $f(x) = \cos(\pi x)$ on the interval $[0,\, 1]$. [Hint: Use the trigonometric identity $2\sin(u)\cos(v) = \sin(u+v) + \sin(u-v)$, if needed.]

8. Using Fourier sine series, find the solution $u(x, t)$ to the time-dependent Schrodinger equation for a free particle in a 1-dimensional box:

$$\begin{cases} \partial_t u = i\partial_{xx} u \\ u(0,t) = 0, u(a,t) = 0\ , \qquad x \in [0, a],\ t \geq 0. \\ u(x,0) \text{ given} \end{cases}$$

(Here, $i = \sqrt{-1}$ is the imaginary constant.) That is, find the Fourier coefficients of the solution in terms of the Fourier coefficients of the initial data $u(x,\, 0)$. Are the modes of the system stable, neutrally stable, or unstable? How does the solution behave and how does this differ from the heat equation studied earlier?