M346 (92153), Sample Midterm #1 Solutions

1. Let $V=\mathbb{R}_2[t]$ with standard basis $\mathcal{B}=\{1,t,t^2\}$ and let $W=M_{2,2}$ be the space of 2×2 real matrices with standard basis $\mathcal{D}=\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$. Consider the linear transformation $L\colon V\to W$ given by

$$L(\boldsymbol{p})=\begin{pmatrix}\boldsymbol{p}(1)-\boldsymbol{p}(0) & \boldsymbol{p}(2)-\boldsymbol{p}(0)\\ \boldsymbol{p}(-1)-\boldsymbol{p}(0) & \boldsymbol{p}(-2)-\boldsymbol{p}(0)\end{pmatrix}.$$

a) Find the matrix representation $[L]_{\mathcal{D}\mathcal{B}}$ of L relative to the bases $\mathcal{B}$ and $\mathcal{D}$.

Solution: Since $L(1)=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, $L(t)=\begin{pmatrix}1&2\\-1&-2\end{pmatrix}$, $L(t^2)=\begin{pmatrix}1&4\\1&4\end{pmatrix}$ the matrix of L is

$$[L]_{\mathcal{D}\mathcal{B}}=\begin{pmatrix}[L(1)]_{\mathcal{D}} & [L(t)]_{\mathcal{D}} & [L(t^2)]_{\mathcal{D}}\end{pmatrix}=\begin{pmatrix}0&1&1\\0&2&4\\0&-1&1\\0&-2&4\end{pmatrix}.$$

b) What is the dimension of Ker(L)? Find a basis for Ker(L).

Solution: The reduced row-echelon form of $[L]_{\mathcal{D}\mathcal{B}}$ is

$$\operatorname{rref}([L]_{\mathcal{D}\mathcal{B}})=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}.$$

There is one column without a pivot, which gives one free variable. Therefore, $\operatorname{nullity}(L):=\dim(\operatorname{Ker}(L))=1$. A basis for the kernel of $[L]_{\mathcal{D}\mathcal{B}}$ is $(1,0,0)^T$, so a basis for Ker(L) is the polynomial $p(t)=1$.

c) What is the dimension of Ran(L)? Find a basis for Ran(L).

Solution: Since there two pivot columns in $\operatorname{rref}([L]_{\mathcal{D}\mathcal{B}})$, $\operatorname{rank}(L):=\dim(\operatorname{Ran}(L))=2$. A basis for the range of $[L]_{\mathcal{D}\mathcal{B}}$ consists of the pivot columns of $[L]_{\mathcal{D}\mathcal{B}}$, i.e., $\{(1,2,-1,-2)^T, (1,4,1,4)^T\}$. This gives the basis $\left\{\begin{pmatrix}1&2\\-1&-2\end{pmatrix},\begin{pmatrix}1&4\\1&4\end{pmatrix}\right\}$ for Ran(L).

2. Let $A=\begin{pmatrix}1&2&3&4\\2&4&5&7\\5&10&13&18\end{pmatrix}$.

a) Let $V=\{\boldsymbol{x}\in\mathbb{R}^4: A\boldsymbol{x}=0\}$. What is the dimension of V? Find a basis for V.

Solution: Since

$$\operatorname{rref}(A)=\begin{pmatrix}1&2&0&1\\0&0&1&1\\0&0&0&0\end{pmatrix},$$

there are two free variables (x_2 and x_4) so $V=\operatorname{Ker}(A)$ has dimension 2. A basis is $\{(-2,1,0,0)^T,(-1,0,-1,1)^T\}$.

b) Are the vectors $(1,2,5)^T$, $(2,4,10)^T$, $(3,5,13)^T$, $(4,7,18)^T$ linearly independent? Do they span $\mathbb{R}^3$?

Solution: The vectors given are the columns of A. Since rref(A) has columns without pivots, the columns of A are not linearly independent. Since rref(A) has a row without a pivot, the columns of A do not span $\mathbb{R}^3$.

c) Give a basis for the span of the four vectors in part (b).

Solution: A basis for the column space of A is given by the pivot columns of A, namely, $(1,2,5)^T$ and $(3,5,13)^T$.

3. Let L: $\mathbb{R}^2 \to \mathbb{R}^2$ be given by $L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8x_1 - 10x_2 \\ 3x_1 - 3x_2 \end{pmatrix}$. Define the standard basis $\mathcal{E} = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and an alternate basis $\mathcal{B} = \left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}\right\}$. Consider a vector $\boldsymbol{v} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$.

a) Find $P_{\mathcal{E}\mathcal{B}}$, $P_{\mathcal{B}\mathcal{E}}$, $[\boldsymbol{v}]_{\mathcal{E}}$, and $[\boldsymbol{v}]_{\mathcal{B}}$.

Solution:

$$P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}, \; P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}, \; [\boldsymbol{v}]_{\mathcal{E}} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}, \; [\boldsymbol{v}]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[\boldsymbol{v}]_{\mathcal{E}} = \begin{pmatrix} 9 \\ -2 \end{pmatrix}.$$

b) Find $[L]_{\mathcal{E}}$ and $[L]_{\mathcal{B}}$.

Solution: $[L]_{\mathcal{E}} = \begin{pmatrix} 8 & -10 \\ 3 & -3 \end{pmatrix}$, $[L]_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}}[L]_{\mathcal{E}}P_{\mathcal{E}\mathcal{B}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

4. True or false?

a) The plane $x_1 + 3x_2 - 4x_3 = 1$ is a subspace of $\mathbb{R}^3$.

Solution: False. The zero element does not lie in the plane so it cannot be a subspace.

b) If A is a 3×5 matrix, then the nullity of A is at least 2.

Solution: True. There are at most 3 pivots in rref(A), so there are at least two columns which give free variables.

c) Let L: $\mathbb{R}_5[t] \to \mathbb{R}^3$ be a linear transformation. If L is onto, the kernel of L has dimension 2.

Solution: False. Since $\operatorname{rank}(L) = \dim(\mathbb{R}^3) = 3$ and $\dim(\mathbb{R}_5[t]) = 6$, by the rank-nullity theorem, $\operatorname{nullity}(L) = \dim(\mathbb{R}_5[t]) - \operatorname{rank}(L) = 6 - 3 = 3$.

d) Let $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be a basis for a vector space V. If n vectors $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n\}$ span V then the coordinate vectors $\{[\boldsymbol{d}_1]_{\mathcal{B}}, \ldots, [\boldsymbol{d}_n]_{\mathcal{B}}\}$ are linearly independent.

Solution: True. Whenever the number of vectors is the same as the dimension of the space, either the vectors both span and are linearly independent, or fail to span and are linearly dependent. Since the vectors $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_n\}$ span V, they are linearly independent in V. Therefore, their coordinate representations $\{[\boldsymbol{d}_1]_{\mathcal{B}}, \ldots, [\boldsymbol{d}_n]_{\mathcal{B}}\}$ are linearly independent in $\mathbb{R}^n$.

e) Every linear transformation $L: \mathbb{R}^5 \to \mathbb{R}^4$ takes the form $L(\boldsymbol{x}) = A\boldsymbol{x}$ with A a 5×4 matrix.

Solution: False. Every linear transformation $L\colon \mathbb{R}^5 \to \mathbb{R}^4$ is given by a 4×5 matrix.

f) Let A_{rref} be the reduced row-echelon form of a matrix A. Then, the pivot columns of A_{rref} form a basis of the column space of A (i.e., the span of the columns of A).

Solution: False. The pivot columns of the *original* matrix A form a basis of the column space of A (i.e., $\operatorname{Ran}(A)$).

g) The vectors $\boldsymbol{b}_1 = 1 + t + 2t^2$, $\boldsymbol{b}_2 = 2 + 3t + 5t^2$, $\boldsymbol{b}_3 = 3 + 7 + 9t^2$ form a basis for $\mathbb{R}_2[t]$.

Solution: True. Using the standard basis, the coordinates of the given vectors are $(1, 1, 2)^T$, $(2, 3, 5)^T$, and $(3, 7, 9)^T$. Since the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 7 \\ 2 & 5 & 9 \end{pmatrix}$ row-reduces to the identity, these vectors form a basis.

h) [Harder...] The equation $p''(t) - p(t) = q(t)$ has a solution $p \in \mathbb{R}_3[t]$ for every $q \in \mathbb{R}_3[t]$.

Solution: True. To see this, we write the equation as $L(p(t)) = q(t)$ where $L = \frac{d^2}{dt^2} - I$ is a linear operator on $\mathbb{R}_3[t]$. The matrix representation of L in the standard basis $\mathcal{B} = \{1, t, t^2, t^3\}$ is

$$[L]_{\mathcal{B}} = \left([L(1)]_{\mathcal{B}} \;\; [L(t)]_{\mathcal{B}} \;\; [L(t^2)]_{\mathcal{B}} \;\; [L(t^3)]_{\mathcal{B}} \right) = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 6 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Since this row-reduces to the identity, we have that $[L]_{\mathcal{B}}$ has full rank and, therefore, L is onto. This immediately implies that $L(p) = q$ has a solution $p \in \mathbb{R}_3[t]$ for every given $q \in \mathbb{R}_3[t]$. Furthermore, the coordinates $[p]_{\mathcal{B}}$ of the solution are found by solving the matrix equation $[L]_{\mathcal{B}}[p]_{\mathcal{B}} = [q]_{\mathcal{B}}$.

5. Consider $A = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$.

a) Write the characteristic polynomial $p_A(\lambda)$ and use this to determine the eigenvalues of A.

Solution: Since $p_A(\lambda) = \det\,(A - \lambda I) = -\lambda\big(\lambda^2 - 1\big) + 2(\lambda - 1) = (\lambda + 2)(\lambda - 1)^2$, A has eigenvalues -2, 1, and 1.

b) Find the eigenspaces corresponding to the eigenvalues of A.

Solution: The corresponding eigenspaces are $E_{-2} = \operatorname{span}\{(1, 1, 1)^T\}$, $E_1 = \operatorname{span}\{(1, -1, 0)^T, (1, 0, -1)^T\}$.

c) Write the matrix in the form $A = PDP^{-1}$ with D a diagonal matrix.

Solution: $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$.