M362K (56310), Problem-solving hints

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List of probability distributions

See pp. 476-488 of the book for a concise summary of named probability distributions (e.g., Bernoulli, Binomial, Normal, Poisson, ...). Properties of and typical examples for these distributions are also given.

Counting

It can be quite confusing to count properly when trying to compute the probability of an event A using an outcome space Ω with equally likely outcomes. Here, we'd like to compute

$$P(A) = \frac{\#(A)}{\#(\Omega)}.$$

There is typically more than one way to specify a proper outcome space for the problem you're asked to solve. The key is to be consistent—it is important to first clarify what a typical outcome $w \in \Omega$ in your outcome space looks like, to count the total number of these outcomes ($\#(\Omega)$), and then to count the number of outcomes corresponding to the event of interest ($\#(A)$). One way to simplify the thought process is to think about the problem entirely in terms of sequences of choices (visualized in terms of a tree diagram, as shown in Appendix 5 of the book). This can be done by stating the procedure in words, then translating it into proper mathematical expressions.

For example, consider the following problem. Suppose there are 100 total tickets that are distributed randomly amongst 10 people, so that each person has 10 tickets. Three of the tickets are winners. Suppose I would like to know the probability that one person wins all three prizes. This can be done as follows. The number of total possible outcomes is

$$\#(\Omega) = \text{“choose three winning tickets out of 100 total tickets”} = \binom{100}{3}.$$

The number of outcomes corresponding to $A =$“one person wins all three prizes” is

$$\begin{aligned}\#(A) &= \text{“choose 1 of 10 people to win all the prizes, then choose 3 out of his 10 tickets to win”} \\ &= \binom{10}{1}\binom{10}{3}.\end{aligned}$$

Each $\binom{\cdot}{\cdot}$ is an unordered choice (i.e., a combination), and multiples of them give sequences of unordered choices. Permutations can also be expressed using this. For example, suppose there is a council with 7 distinct positions (president, vice-president, ...), for which 15 men and 17 women are candidates. How many ways are there to have a council with 3 men and 4 women?

First we choose 3 out of the 15 men to serve on the council, then choose 4 out of the 17 women to serve on the council, then choose 1 of the 7 total selected people to be president, then choose 1 of the 6 remaining to be vice-president, ... This gives us

$$\binom{15}{3}\binom{17}{4}\binom{7}{1}\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \binom{15}{3}\binom{17}{4}7!$$

different ways. Now suppose we ask the same thing, but require that the president be a woman and the vice-president a man. To do this, we first choose 3 out of the 15 men to serve on the council, then choose 4 out of the 17 women to serve on the council, then choose 1 of the 4 selected women to be president, then choose 1 of the 3 selected men to be vice-president, then choose 1 of the 5 remaining selected people to be secretary, 1 of the 4 remaining to be treasurer, ...:

$$\binom{15}{3}\binom{17}{4}\binom{4}{1}\binom{3}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = \binom{15}{3}\binom{17}{4} 4 \cdot 3 \cdot 5!$$

We can also express multinomials easily. Going back to the example of making 11-letter words from MISSISSIPPI, we first choose 1 spot out of the 11 for the M, then choose 4 of the remaining 10 spots for the I's, then choose 4 of the remaining 6 spots for the S's, then choose 23 of the remaining 2 spots for the P's:

$$\binom{11}{1}\binom{10}{4}\binom{6}{4}\binom{2}{2}.$$

There's nothing special about making the choices of spots this way—we can change the order of selection, first choosing the S's, then the P's, then the M, and lastly the I's. The end result is always the same, and is the *multinomial coefficient* corresponding to breaking the set of 11 spots into 4 unordered groups of size 1, 4, 4, and 2:

$$\binom{11}{1\ \ 4\ \ 4\ \ 2}.$$

Now suppose we want to count the number of 6 letter words using the letters MISSISSIPPI. Can you think of a counting procedure to find the answer?

Decomposing events into simpler events

There are many problems in which is is easier to compute the probability of the complement of an event rather than the probability of the event itself. In these cases, we use the complement rule $P(A) = 1 - P(A^c)$. For example, suppose we flip a fair coin ten times and ask for the probability of getting at least one head—that is, we seek $P(\geq 1 \text{ heads})$. Since the event of getting no heads is the complement of this event, we have that

$$P(\geq 1 \text{ heads}) = 1 - P(\text{exactly } 0 \text{ heads}) = 1 - \left(\frac{1}{2}\right)^{10}.$$

Similarly, for probability of multiple events or conditional probabilities we can use the rules

$$P(A \cap B) = 1 - P(A^c \cap B)$$
$$P(A|C) = 1 - P(A^c|C)$$

for any A, B, and C.

Another useful trick is to keep in mind that for any random variable X which takes on only integer values,

$$P(X = k) = P(X \leq k) - P(X \leq k-1) = P(X < k+1) - P(X < k).$$

Similarly,

$$P(X = k) = P(X \geq k) - P(X \geq k+1) = P(X > k-1) - P(X > k)$$

This is useful since it is sometimes easier to compute probabilities of events $\{X \leq k\}$ or $\{X \geq k\}$ instead of the event $\{X = k\}$.