M408D (54690/95/00), Final Exam Solutions: Part 1, 12/13/2010

Multiple choice questions #1.1-1.4 (20 points)

See last two pages of solutions.

Question #1.1 (20 points)

Define the function

$$f(x,t) = t^{-1/2} e^{-x^2/t}, \qquad t > 0.$$

a) Determine $\partial f/\partial t$ and $\partial f/\partial x$.

Solution:

$$\frac{\partial f}{\partial x} = -\frac{2x}{t^{3/2}} e^{-x^2/t}, \qquad \frac{\partial f}{\partial t} = \left(\frac{x^2}{t^{5/2}} - \frac{1}{2t^{3/2}}\right) e^{-x^2/t}$$

b) Consider the partial differential equation (PDE)

$$\frac{\partial f}{\partial t} = \frac{1}{4}\frac{\partial^2 f}{\partial x^2},$$

known as the *heat equation* since it describes the flow of heat in a thin tube. Show that $f(x,t)$ as defined above a solution to this PDE (i.e., verify that it satisfies the equation).

Solution: Since

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{2x}{t^{3/2}} e^{-x^2/t}\right) = \left(\frac{4x^2}{t^{5/2}} - \frac{2}{t^{3/2}}\right) e^{-x^2/t} = 4\frac{\partial f}{\partial t},$$

f satisfies the heat equation.

c) Let $\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{j}$ be the direction of a unit vector $\boldsymbol{u}$ in the xt-plane. Find the directional derivative $D_{\boldsymbol{u}} f$ at the point $P(0,1)$.

[Hint: Use part (a).]

Solution: In part (a) we have computed the gradient

$$\nabla f = \langle f_x, f_t \rangle = -\frac{2x}{t^{3/2}} e^{-x^2/t} \boldsymbol{i} + \left(\frac{x^2}{t^{5/2}} - \frac{1}{2t^{3/2}}\right) e^{-x^2/t} \boldsymbol{j}.$$

Therefore, with $\boldsymbol{u} = \frac{1}{\sqrt{2}}(\boldsymbol{i} - \boldsymbol{j})$ the unit vector in the direction of $\boldsymbol{v}$, the directional derivative at $P(0,1)$ is

$$D_{\boldsymbol{u}} f\Big|_{(0,1)} = \frac{1}{\sqrt{2}} \nabla f \cdot \langle 1, -1 \rangle \Big|_{(0,1)} = \frac{1}{2\sqrt{2}}.$$

d) Now let $x(s,v,w) = svw$ and $t(s,v) = v + \sin(2s)$. Determine $\partial f/\partial s$ at $s = 0$.

Solution: First note that $\frac{\partial x}{\partial s} = vw$ and $\frac{\partial t}{\partial s} = 2\cos(2s)$. Using the chain rule, we find

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial s} = -\frac{2vwx}{t^{3/2}} e^{-x^2/t} + 2\cos(2s)\left(\frac{x^2}{t^{5/2}} - \frac{1}{2t^{3/2}}\right) e^{-x^2/t}.$$

Since $x|_{s=0} = 0$ and $t\big|_{s=0} = v$, we get that $\partial f/\partial s\big|_{s=0} = -v^{-3/2}$.

Question #1.2 (20 points)

Define

$$f(x, y) = e^y(y^2 - x^2).$$

a) Find the critical points of f.

Solution: First we note that $\nabla f = \langle -2xe^y, (y^2 - x^2 + 2y)e^y \rangle$. The critical points of f satisfy $\nabla f = 0$, so that $-2x = (y^2 - x^2 + 2y) = 0$. This implies that $x = 0$ and that $(0, 0)$ and $(0, -2)$ are the only critical points of f.

b) Compute the Hessian

$$H(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

of the function. Then classify all critical points by using the second derivative test (i.e., determine if they are local maxima, minima, or saddle points).

Solution: Since $f_{xx} = -2e^y$, $f_{xy} = f_{yx} = -2xe^y$, $f_{yy} = (y^2 - x^2 + 4y + 2)e^y$. Then $D(0, 0) = \det H(0, 0) = -2 \cdot 2 - 0^2 = -4 < 0$ so $(0, 0)$ is a saddle point. However, $D(0, -2) = \det H(0, -2) = (-2e^{-2})(-2e^{-2}) - 0^2 = 4e^{-4} > 0$. Since $f_{xx}(0, -2) = -2e^{-2} < 0$, $(0, -2)$ is a local maximum.

c) What is the absolute maximum value of f on $D = \{(x, y): 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$?

Solution: Since f is continuous and has no local maxima in the interior of D, it must achieve its absolute maximum on the boundary of D. Testing f on each of the four boundaries of the square D, we find that the absolute maximum is e (achieved at $(0, 1)$).

Question #1.3 (20 points)

Evaluate the following integrals. Remember that iterated integrals are sometimes easier to evaluate after switching the order of integration, or after changing to different coordinates.

a) Determine

$$\iint_R (4 + x^2 - y^2)\, dA$$

where the region of integration is the rectangle $R = \{(x, y): -1 \leq x \leq 1,\ 0 \leq y \leq 2\}$.

Solution: Writing this integral with the proper boundaries of integration we find that

$$\begin{aligned} \int_{-1}^{1} \int_0^2 (4 + x^2 - y^2)\, dy\, dx &= \int_{-1}^{1} \left[4y + x^2 y - \frac{1}{3} y^3 \right]_{y=0}^{y=2} dx \\ &= \int_{-1}^{1} \left(2x^2 + \frac{16}{3} \right) dx \\ &= \left[\frac{2}{3} x^3 + \frac{16}{3} x \right]_{-1}^{1} = 12. \end{aligned}$$

b) Consider the integral

$$\iint_D 2x^2 e^{xy} dA$$

over the triangular region $D = \{(x, y): 0 \leq x \leq 2,\ 0 \leq y \leq x\}$. Provide **expressions** for the two possible iterated integrals (one integrating in x first, the other in y first) with correct boundaries of integration. Do not evaluate these yet.

Solution: The two (equivalent) iterated integrals are

$$\int_0^2 \int_y^2 2x^2 e^{x\,y} dx\,dy, \qquad \int_0^2 \int_0^x 2x^2 e^{x\,y} dy\,dx.$$

c) Evaluate the integral in part (b) using one of the two iterated integrals.

Solution: It is easier to evaluate the integral by first integrating in y, then in x. Then,

$$\begin{aligned}\int_0^2 \int_0^x 2x^2 e^{x\,y} dy\,dx &= \int_0^2 [2x\,e^{x\,y}]_{y=0}^{y=x}\,dx \\ &= \int_0^2 (2x\,e^{x^2} - 2x)dx \\ &= \int_0^4 e^u du - 4 \\ &= e^4 - 5.\end{aligned}$$

Question #1.4 (20 points)

We will evaluate the double integral

$$I = \int\int_R \sin\left(\pi(9x^2 + 4y^2)\right)\,dA$$

over the region R bounded by the the ellipse $9x^2 + 4y^2 = 1$ by completing the following sequence of steps.

a) Consider the linear transformation $T: (u, v) \to (x, y)$ given by

$$x = \frac{1}{3}u, \qquad y = \frac{1}{2}v.$$

Find the Jacobian $J = \frac{\partial(x,\,y)}{\partial(u,\,v)}$ of the transformation.

Solution:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{pmatrix} = \det\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2}\end{pmatrix} = \frac{1}{6}.$$

b) After the change of variables to (u, v) coordinates, the integral I becomes

$$\int\int_S \sin\left(\pi(u^2 + v^2)\right)|J|du\,dv$$

with $S = \{(u, v)\colon u^2 + v^2 \leq 1\}$. Transform to polar coordinates to express this as an iterated integral in terms of the variables r and θ.

Solution: With $r^2 = u^2 + v^2$ we write the integral as

$$I = \int_0^{2\pi} \int_0^1 \frac{1}{6}\sin\left(\pi r^2\right) r\,dr\,d\theta.$$

c) Evaluate the answer obtained in (b).

Solution: With the substitution $w = r^2$ we find that

$$I = \frac{\pi}{6}\int_0^1 \sin(\pi u)du = -\frac{1}{6}[\cos(\pi u)]_0^1 = \frac{1}{3}.$$

Multiple choice questions #2.1-2.4 (*0 points*)

Not included in exam (failed to print).

Question #2.1 (20 points)

Determine if the following series is absolutely convergent, conditionally convergent, or divergent. **You must show your work and justify the use of any test to obtain credit.**

a)

$$\sum_{n=1}^{\infty} \frac{e^n}{n^2}$$

[Hint: Consider $\lim_{n\to\infty} e^n/n^2$.]

Solution: By L'Hospital's rule applied twice,

$$\lim_{n\to\infty} e^x/x^2 = \lim_{n\to\infty} e^x/2x = \lim_{n\to\infty} e^x/2 = +\infty.$$

Therefore, by the divergence test we have that the series diverges (since the sequence being summed does not go to zero as $n \to 0$).

b)

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^{2/3} + n^{2/3}\ln n}$$

Solution: This problem is nearly identical to that from the first midterm. First we test for absolute convergence of the series. We must find whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3} + n^{2/3}\ln n}$$

converges or not. To do so, note that the first term in the denominator grows faster than the second term and is therefore the dominant contribution as $n \to \infty$. This can be seen by noting that

$$\lim_{n\to\infty} \frac{n^{2/3}\ln n}{n(\ln n)^{2/3}} = \left(\lim_{n\to\infty} \frac{\ln n}{n}\right)^{1/3} = 0$$

by L'Hospital's rule. Therefore, by the limit comparison test we have that the series converges/diverges if the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3}}$$

converges/diverges. Now note that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{2/3}}$ diverges by the integral test since the improper integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^{2/3}} dx = \int_{\ln 2}^{\infty} \frac{1}{u^{2/3}} du \qquad \text{(substitution with } u = \ln x\text{)}$$

diverges. So $\sum_{n=2}^{\infty} \frac{1}{n\,(\ln n)^{2/3}}$ diverges and the original series is *not* absolutely convergent.

[There are many other ways to solve this problem by comparing against a series which we know is divergent. For example, since $\frac{1}{n\,(\ln n)^{2/3}+n^{2/3}\ln n} \geq \frac{1}{2n\ln n}$ and $\sum \frac{1}{2n\ln n}$ diverges by the integral test, we find that the original series is not absolutely convergent.]

Now we show that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\,(\ln n)^{2/3}+n^{2/3}\ln n}$$

is in fact conditionally convergent by the alternating series test. Indeed, it is simple to check that

$$b_n = \frac{1}{n\,(\ln n)^{2/3}+n^{2/3}\ln n}$$

satisfies $\lim_{n\to\infty} b_n = 0$ and $\{b_n\}$ is a decreasing sequence of terms. We therefore conclude that the series is conditionally convergent.

Question #2.2 (20 points)

a) Determine the *interval of convergence* of the following power series centered at $a=1$:

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n\sqrt{n}}$$

[Hint: Start by using the ratio or root test to find the radius of convergence of the series.]

Solution: By the ratio test, we find that the series is absolutely convergent when

$$\lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(x-1)^n}\cdot\frac{2^n\sqrt{n}}{2^{n+1}\sqrt{n+1}}\right| = \frac{1}{2}|x-1|\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = \frac{1}{2}|x-1| < 1.$$

So the radius of convergence of the series is 2 and the interval of convergence is either $(-1,\,3)$, $[-1,\,3)$, $(-1,\,3]$, or $[-1,\,3]$. Testing the endpoints we see that $x=-1$ is in the interval of convergence using the alternating series test, but $x=3$ is not using the p-series test with $p=1/2$. So $I=[-1,3)$.

b) Find the second-degree polynomial $T_2(x)$ that best approximates the function $f(x)=e^{-x^2}$ **near $x=1$**.

Solution: We must find the second-order Taylor polynomial centered at $a=1$. Since

$$f'(x) = -2x\,e^{-x^2}, \qquad f''(x) = (4x^2-2)\,e^{-x^2}$$

we have that

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{1}{2!}f''(1)(x-1)^2 = e^{-1}\big(1-2(x-1)+(x-1)^2\big).$$

c) Derive the Maclaurin series (i.e., Taylor series centered at 0) of e^{-x^2}. What is the radius of convergence of the series?

Solution: Finding the nth derivative of e^{-x^2} is not particularly easy, so instead we use the known Maclaurin series for e^x. Since

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \qquad -\infty < z < \infty,$$

making the substitution $z = -x^2$ we have that

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}, \qquad -\infty < x < \infty.$$

The radius of convergence of the series is infinite (series converges everywhere).

Question #2.3 (20 points)

Define the vectors

$$\begin{aligned} \boldsymbol{u} &= \langle 4, 5, -1 \rangle \\ \boldsymbol{v} &= \langle 1, 0, 1 \rangle \\ \boldsymbol{w} &= \langle 3, -2, 2 \rangle. \end{aligned}$$

a) Compute $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$.

Solution: First we compute $\boldsymbol{v} \times \boldsymbol{w}$:

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & 1 \\ 3 & -2 & 2 \end{vmatrix} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}.$$

Therefore, $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = 8 + 5 + 2 = 15$.

b) Determine the vector equation for the plane parallel to the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ that passes through the point $P(0, 1, 2)$ (i.e., express in the form $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$ by determining a normal vector $\boldsymbol{n}$ and an intercept $\boldsymbol{r_0}$).

Solution: The plane parallel to $\boldsymbol{v}$ and $\boldsymbol{w}$ has normal vector $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle 2, 1, -2 \rangle$. Since it passes through $P(0, 1, 2)$, the intercept is $\boldsymbol{r_0} = \langle 0, 1, 2 \rangle$.

c) Find the shortest distance between the point $P(0, 1, 2)$ and the plane given by

$$x + 3y + 5z = 48$$

using the method of Lagrange multipliers. At the least, write down the proper equations to be solved.

[Hint: Need to use equations $\nabla f = \lambda \nabla g$, $g = \text{const.}$, for an appropriate choice of function f to be extremized under some constraint g.]

Solution: The distance between the point $P(0, 1, 2)$ and any other point $Q(x, y, z)$ is

$$d(x, y, z) = \sqrt{x^2 + (y-1)^2 + (z-2)^2}.$$

Therefore, we seek to minimize $d(x, y, z)$ subject to the constraint

$$g(x, y, z) = x + 3y + 5z = 48$$

(which simply says that Q must lie on the given plane). To make things simpler, note that we can, equivalently, minimize one-half the squared distance

$$f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}(y-1)^2 + \frac{1}{2}(z-2)^2$$

subject to $g(x, y, z) = 48$. We minimize this using Lagrange multipliers. That is, we need to solve the system of equations $\nabla f = \lambda \nabla g,\ g = 48$. This yields the system of equations

$$\begin{aligned} x &= \lambda \\ y &= 1 + 3\lambda \\ z &= 2 + 5\lambda \\ x + 3y + 5z &= 48. \end{aligned}$$

Substituting x, y, z in terms of λ into the last equation we find that $13 + 35\lambda = 48$, i.e., $\lambda = 1$ and $(x, y, z) = (1, 4, 7)$. Is is then simple to check that this corresponds to a minimal distance of $d(1, 4, 7) = \sqrt{35}$ between P and the plane.

Question #2.4 (20 points)

Define the vector-valued function $\boldsymbol{r}(t)$ through its derivative

$$\boldsymbol{r}'(t) = \langle \cos t, -\sin t, 0 \rangle$$

and suppose $\boldsymbol{r}(0) = \langle 1, 1, 1 \rangle$.

a) Find $\boldsymbol{r}(t)$.

Solution: Integrating, we have that

$$\boldsymbol{r}(t) = \boldsymbol{r}(0) + \int_0^t \boldsymbol{r}'(s)\, ds = \langle 1 + \sin t, \cos t, 0 \rangle.$$

b) Find the unit tangent vector $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$.

Solution: Since $|\boldsymbol{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1$, we find that $\boldsymbol{T}(t) = \boldsymbol{r}'(t) = \langle \cos t, -\sin t, 0 \rangle$.

c) The parametric curve traced by $\boldsymbol{r}(t)$ lies on the surface of the paraboloid

$$z = x^2 + (y-1)^2 - c$$

for which value of c?

Solution: The parametric curve traced out by $\boldsymbol{r}(t)$ is a circle of radius 1 in the xy-plane centered at $(1, 0)$. Therefore, the curve lies on the surface $(x-1)^2 + y^2 = 1$, implying that

$$c = 2(x - y) + 1 = 2(\sin t - \cos t) + 3.$$

Note that there was an error in the statement of the question that made the problem more complicated—the equation for the paraboloid was originally meant to read "$z = (x-1)^2 + y^2 - c$," in which case the answer was $c = 1$. Regardless, it was possible to solve for c by substituting the coordinates of $\mathbf{r}(t)$ for x, y, and z in the given equation.

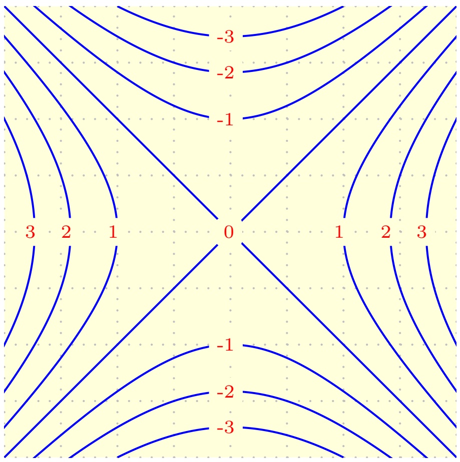
d) Find the arc length of one loop of the parametric curve, either by using the arc length formula $L = \int_a^b |\boldsymbol{r}'(t)| dt$ or by using a more direct method.

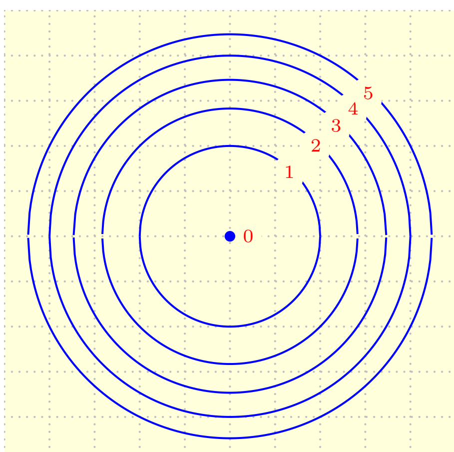
Solution: The curve returns to its starting position each time t increases by 2π. Using the arc length formula we find $L = \int_0^{2\pi} 1\, dt = 2\pi$. More simply, this is found using the formula for the length of a circle with radius 1.

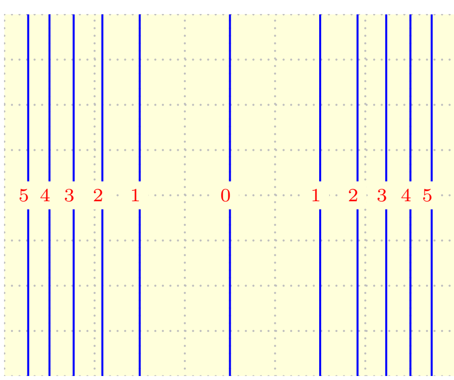
This print-out should have 4 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

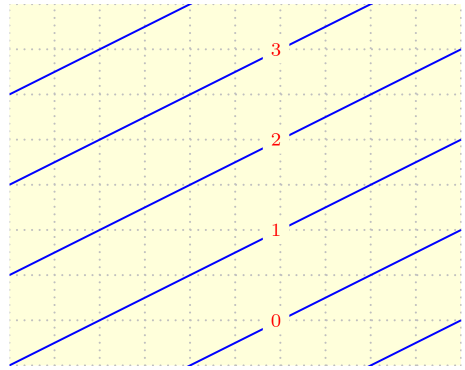
CalC15a20b
001 5.0 points

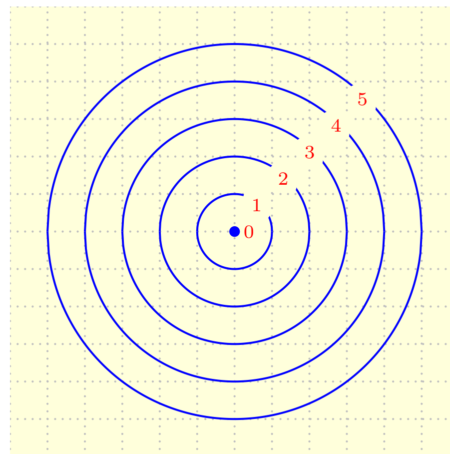
Which one of the following could be the contour map of a cone?

1. 

2. 

3. 

4. 

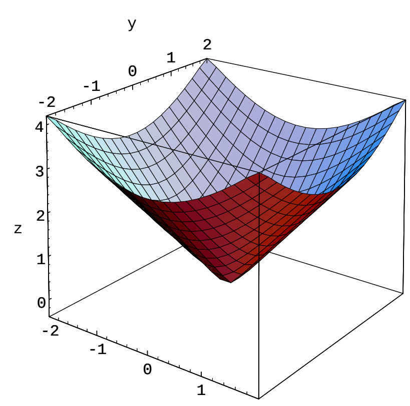
5. 

correct

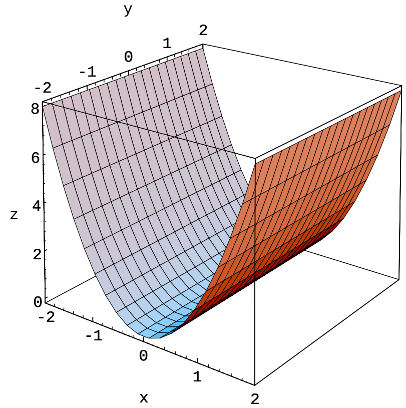
CalC15a30a
002 5.0 points

Which one of the following surfaces is the graph of

$$f(x, y) = 2x^2 \ ?$$

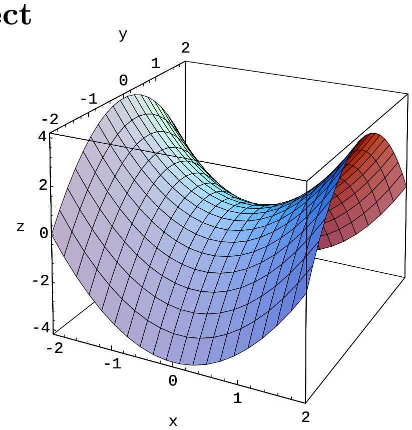
1. 

2.

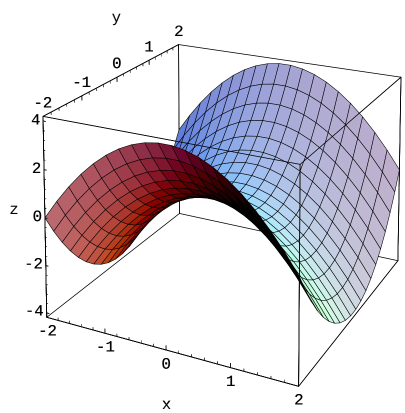


correct

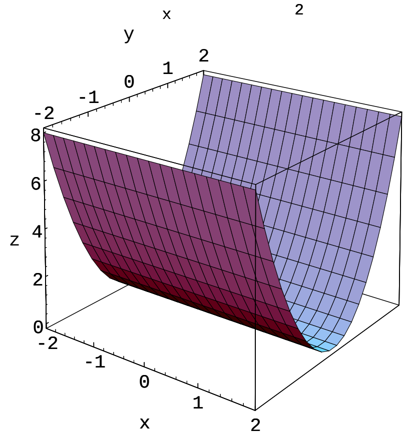
3.



4.



5.



CalC15f23s
003 5.0 points

Find the maximum slope on the graph of

$$f(x, y) = 3\sin(xy)$$

at the point $P(0, 4)$.

1. max slope $= 12$ **correct**

2. max slope $= 4\pi$

3. max slope $= \pi$

4. max slope $= 3$

5. max slope $= 1$

6. max slope $= 12\pi$

7. max slope $= 3\pi$

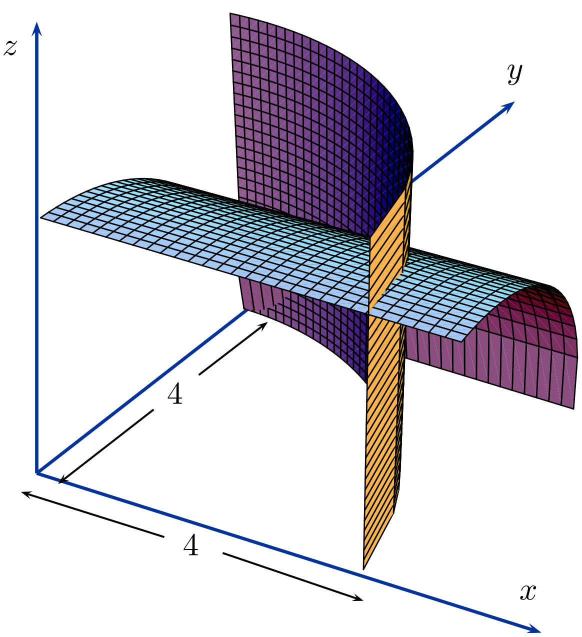
8. max slope $= 4$

CalC16c28s
004 5.0 points

Find the volume of the solid in the first octant bounded by the cylinders

$$x^2 + y^2 = 16, \qquad y^2 + z^2 = 16.$$

Hint: in the first octant the cylinders are shown in



1. volume $= \frac{112}{3}$ cu. units

2. volume $= \dfrac{128}{3}$ cu. units **correct**

3. volume $= 40$ cu. units

4. volume $= \dfrac{116}{3}$ cu. units

5. volume $= \dfrac{124}{3}$ cu. units