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08/26/10

Limits (2.2)

Recall definition of limit:

$$\lim_{x \to a} f(x) = L \quad (f(x) \to L \text{ as } x \to a)$$

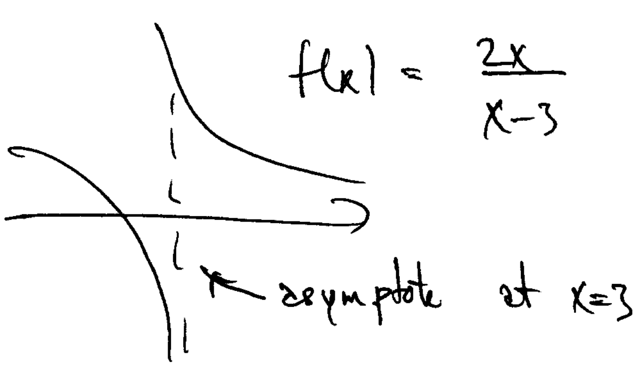
$\Leftrightarrow$ $f(x)$ as close to L as we like by choosing x close (but not equal to a), coming from either side.

Holds for $a = +\infty$ or $-\infty$, or $L = \pm\infty$ as well.

Sometimes we'll have to consider left- or right-handed limits, if the limit above does not exist:

$$\lim_{x \to a^-} f(x) = L \quad \text{or} \quad \lim_{x \to a^+} f(x) = L$$

Ex. $f(x) = \dfrac{2x}{x-3}$



$$\lim_{x \to 3^+} f(x) = +\infty$$

$$\lim_{x \to 3^-} f(x) = -\infty$$

See (2.3), p. 77 for limit laws such as

$$\lim_{x\to a} [f(x)\,g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right).$$

if each limit exists.

Indeterminate forms and L'Hospital's rule

① Suppose we have two functions $f(x)$, $g(x)$, such that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.

Then $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0}$$? what does this mean?

Ex. $$\lim_{x\to 2} \frac{7x-14}{x^2-4} = \frac{\lim_{x\to 2}(7x-14)}{\lim_{x\to 2}(x^2-4)} = \frac{0}{0} \; ?$$

$$\lim_{x\to 2} \frac{7\cancel{(x-2)}}{(x+2)\cancel{(x-2)}} = \lim_{x\to 2} \frac{7}{x+2} = \boxed{\frac{7}{4}}.$$

Simplification by factorization, then take limits.

What if this doesn't work?

Ex. $\displaystyle\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = ?$

② Now suppose $\displaystyle\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and
$\displaystyle\lim_{x \to a} g(x) = \infty$ (or $-\infty$).

Then $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{\infty}{\infty} ?$

Ex. $\displaystyle\lim_{x \to \infty} \frac{3x^3 + 7x}{2x^3 - 3} = \lim_{x \to \infty} \frac{3 + \frac{7}{x^2}}{2 - \frac{3}{x^3}}$

$= \boxed{\frac{3}{2}}$.

Ex. $\displaystyle\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = ?$

L'Hopital's rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ in a small interval containing a (except possibly at a). If we have a limit of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Ex. $$\lim_{x\to 0}\frac{\overbrace{e^x-1-x}^{f(x)}}{\underbrace{x^2}_{g(x)}} \overset{\star}{=} \lim_{x\to 0}\frac{\overbrace{e^x-1}^{\tilde{f}(x)}}{\underbrace{2x}_{\tilde{g}(x)}} \overset{\star\star}{=} \lim_{x\to 0}\frac{e^x}{2} = \boxed{\frac{1}{2}}.$$

$\star$ Check that of form $\frac{0}{0}$.

$f'(x) = e^x - 1$, $g'(x) = 2x$

$\star\star$ Check that of form $\frac{0}{0}$

$\tilde{f}'(x) = e^x$, $\tilde{g}'(x) = 2$

Ex. $$\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}} \overset{\star}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}} = \lim_{x\to\infty}\frac{3x^{2/3}}{x} = \lim_{x\to\infty}\frac{3}{x^{1/3}} = \boxed{0}$$

$\star$ Check that of form $\frac{\infty}{\infty}$

$f'(x) = \frac{1}{x}$, $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

__Warning:__ L'Hospital's rule doesn't work __unless__ you're working with $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form!

Ex. $$\lim_{x\to\pi^-}\frac{\sin x}{1-\cos x} \neq \lim_{x\to\pi^-}\frac{\cos(x)}{\sin(x)} = -\infty$$

$$\frac{\lim_{x\to\pi^-} \sin x}{\lim_{x\to\pi^-} (1-\cos x)} = \frac{0}{2} = \boxed{0}.$$

Indeterminate products:

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$

then what is $\lim_{x\to a} f(x)\,g(x)$?

Idea: Convert to indeterminate ratio.

Ex. $\lim_{x\to\infty} \underbrace{x}_{g(x)} \underbrace{\sin\left(\frac{\pi}{x}\right)}_{f(x)}$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} \qquad \left(\text{of form } \frac{0}{0}\right)$$

$$= \lim_{y\to 0^+} \frac{\sin(\pi y)}{y} \qquad y = \frac{1}{x}$$

$$\stackrel{\text{L'Hospital's rule.}}{=} \lim_{y\to 0^+} \frac{\pi\cos(\pi y)}{1} = \boxed{\pi}.$$

$$= \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\left(\pi \cdot \cancel{\frac{-1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}}$$

$$= \boxed{\pi}.$$

Indeterminate differences:

$\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = \infty$

What is $\lim_{x\to a} [f(x) - g(x)] = ?$

$\boxed{\text{of form } \infty - \infty}$

Idea: Factor to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

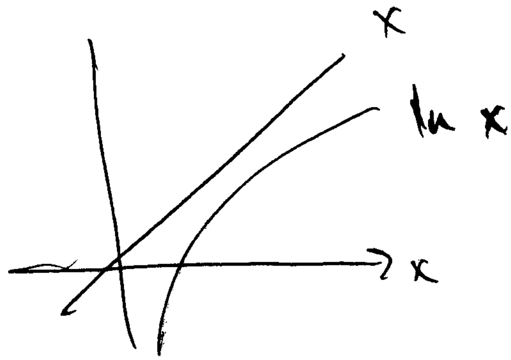
Ex. $\lim_{x\to\infty} (x - \ln x)$

$$= \lim_{x\to\infty} x\left(1 - \frac{\ln x}{x}\right) = \infty \cdot 1 = \boxed{\infty.}$$

$\frac{\ln x}{x} \to 0$ as $x\to\infty$

Q: What is $\lim_{x\to\infty} \frac{\ln x}{x}$?

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$



Indeterminate powers: What is $\lim_{x\to a} f(x)^{g(x)}$?

if i) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

$$\boxed{\text{form } 0^0}$$

ii) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$

$$\boxed{\text{form } \infty^0}$$

iii) $\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} g(x) = \pm\infty$

$$\boxed{\text{form } 1^\infty}$$

Idea: Take natural logarithm to get indeterminate product.

Ex. $\lim_{x\to 0^+} x^x = ?$ $\boxed{\text{of form } 0^0}$

Let $y = x^x$, so $\ln y = \ln(x^x)$
$= x\ln(x)$

$\Rightarrow$ $y = e^{x\ln x}$.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x}$$
$$= e^{\left(\lim_{x\to 0^+} x\ln x\right)} = e^0 = \boxed{1}.$$

$\longrightarrow \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}}$ (form $\frac{-\infty}{\infty}$)

(form $0\cdot -\infty$)

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$
$$= \lim_{x\to 0^+} (-x) = 0.$$

Ex. 9, p. 476:

$$\lim_{x\to 0^+} (\underbrace{1+\sin 4x}_{1})^{\overbrace{\cot x}^{+\infty}}$$

$\boxed{1^\infty}$

a^b

$y = a^b = e^{b\log a}$

$\ln y = \ln(a^b)$
$= b\ln(a)$

$\cot x = \dfrac{\cos x}{\sin x}$

$$= \lim_{x\to 0^+} \exp\left(\cancel{\ } (\cot x) \ln(1+\sin 4x) \right)$$

$$= \exp\left(\underbrace{\lim_{x\to 0^+} \overbrace{(\cot x)}^{+\infty} \overbrace{\ln(1+\sin 4x)}^{0}} \right)$$

$e^\wedge$

$\boxed{0\cdot\infty}$ $= ?$

$$\lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan(x)}$$

$\sec(x) = \dfrac{1}{\cos(x)}$

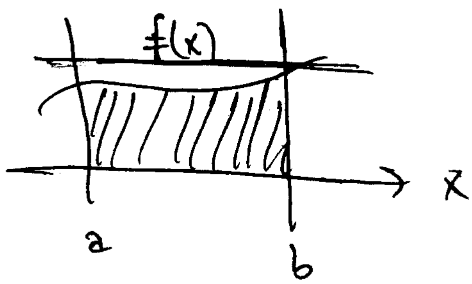
$$= \lim_{x\to 0^+} \frac{\frac{1}{1+\sin(4x)} \cdot 4\cos(4x)}{\sec^2(x)}$$

$$= 4.$$

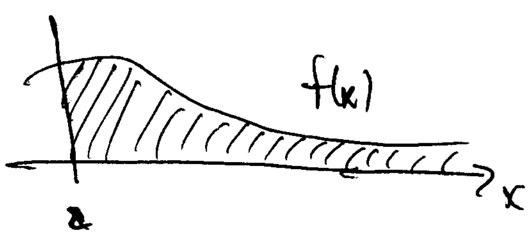
$\Rightarrow$ $\boxed{\text{Answer: } e^4}$.

Improper integrals (8.8)

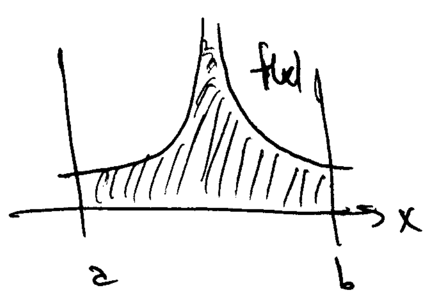
Up until now, have dealt with integrals $\int_a^b f(x)\,dx$, where $[a,b]$ is a finite interval and f is a bounded function:



What if ① $[a,b]$ is an infinite interval, i.e, $a=-\infty$ or $b=+\infty$, or both?

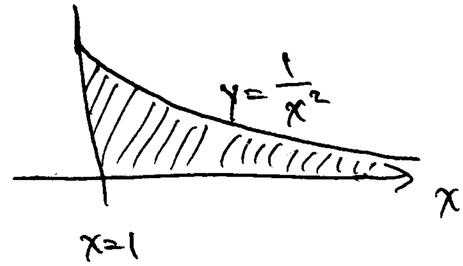


② $f \geq 0$ and f is unbounded, i.e, f has an infinite discontinuity?



① Infinite interval:

Ex. Find area under $y = \frac{1}{x^2}$ to the right of $x=1$, i.e., $\int_1^\infty \frac{1}{x^2}\,dx$.



Let $A(t)$, $t \geq 1$ be the area under the graph between $x=1$ and $x=t$:

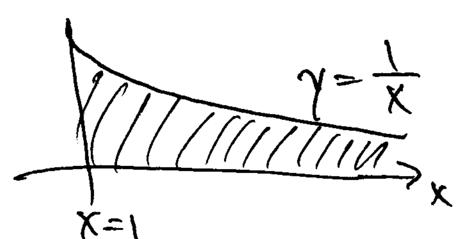
$$A(t) = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^t = 1 - \frac{1}{t}, \quad t \geq 1$$

So, we define $\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} A(t) = 1$.

Convergent!

Bad things can happen though ...

Ex. Find area to the right of $x=1$ and under $y = \frac{1}{x}$.



$$A(t) = \int_1^t \frac{1}{x}\,dx = \ln x \Big|_1^t = \ln t - \cancel{\ln 1}^{\,0}$$

divergent!

So, $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} A(t) = \lim_{t\to\infty} \ln t = \infty$!

Def. (a) If $\int_a^t f(x)\,dx$ exists for every $t \geq a$ and $\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists as a finite number, then define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx .$$

In this case, the improper integral $\int_a^\infty f(x)\,dx$ is convergent. If $\lim_{t\to\infty} \int_a^t f(x)\,dx$ does not exist, the improper integral is divergent.

(b) Similar definition for $\int_{-\infty}^b f(x)\,dx$:

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to -\infty} \int_t^b f(x)\,dx ,$$

if limit exists and is finite.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then define

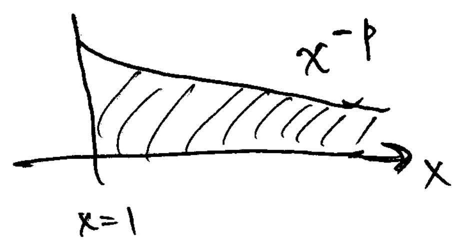
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

(with a any real number).

<u>Remark</u>: $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent while $\int_1^\infty \frac{1}{x}\,dx$ is divergent since $\frac{1}{x^2}$ goes to zero faster than $\frac{1}{x}$ as $x \to \infty$.

<u>Moral</u>: It is possible to add an infinite no. of quantities and still get something finite!

<u>Ex.</u> For what p is $\int_1^\infty \frac{1}{x^p}\,dx$ convergent?



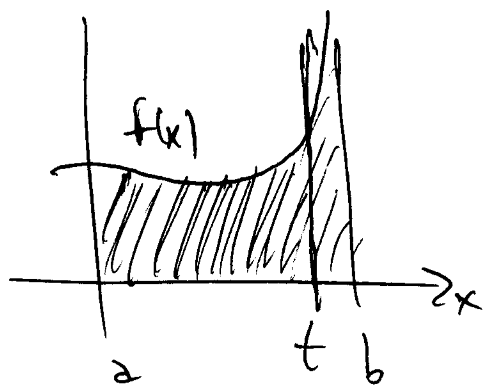
Assume $p \neq 1$.

Then
$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$
$$= \lim_{t\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right]_1^t$$
$$= \lim_{t\to\infty} \frac{1}{1-p}\left(\frac{1}{t^{p-1}} - 1\right)$$

$$\Rightarrow \boxed{\int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{p-1} \quad \text{if } p>1 \quad \text{and is divergent if } p \le 1.}$$

② Unbounded integrands (infinite discontinuity):

Suppose $f \geq 0$ and is continuous in $[a,b)$ but has a vertical asymptote at $x=b$:



<u>Def.</u> (a) If f continuous on $[a,b)$ and has infinite discontinuity at $x=b$, and

$\lim_{t\to b^-} \int_a^t f(x)\,dx$ <u>exists as a finite number</u>

then define

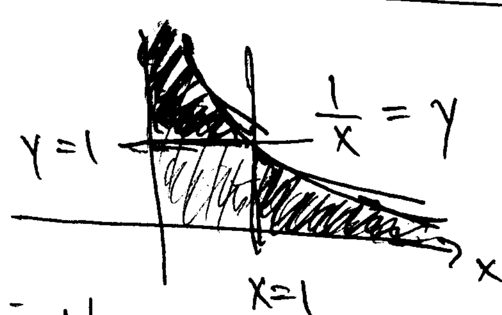
$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

$$= \lim_{t\to 0^+} \frac{x^{-p+1}}{-p+1}\Bigg]_{t=x}^{1=x}$$

$$= \lim_{t\to 0^+} \frac{1}{1-p}\left(1 - t^{1-p}\right)$$

$$\Rightarrow \boxed{\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p} \text{ if } p<1 \text{ and is divergent if } p \geq 1}$$

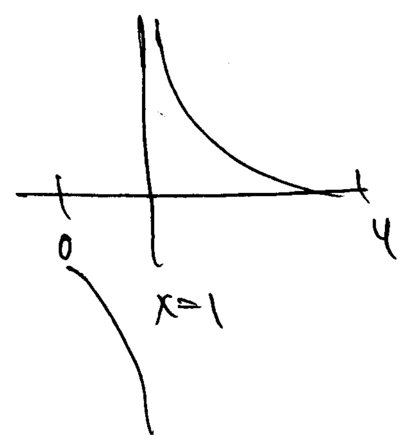
When $p=1$:



Ex. $\int_0^4 \frac{dx}{x-1}$ divergent! $\rightsquigarrow$ $\frac{1}{x-1}$ has an infinite discont. at $x=1$.

$$\| $$

$$\int_0^1 \frac{1}{x-1}\,dx + \int_1^4 \frac{1}{x-1}\,dx$$



$$\int_0^1 \frac{1}{x-1}\,dx = \lim_{t\to 1^-} \int_0^t \frac{1}{x-1}\,dx$$

$$= \lim_{t\to 1^-} \ln|x-1|\Big]_{x=0}^{x=t} = \lim_{t\to 1^-} \ln(1-t)$$

$\boxed{\text{divergent!}}$ $-\infty$!

Warning: $\displaystyle\int_0^4 \frac{dx}{x-1} = \ln|x-1|\Big]_{x=0}^{x=4}$

$= \ln 3 - \cancel{\ln 1}^{\,0}$

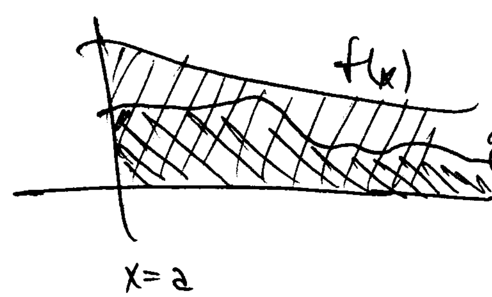
$= \ln 3.$

$\boxed{\text{Wrong!}}$

Comparison test for improper integrals

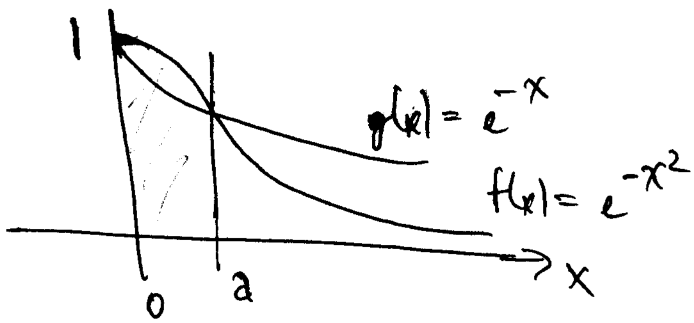
Thm. Suppose f and g are both positive, continuous functions with $f(x) \geq g(x) \geq 0$ for $x = a$.

(a) If $\displaystyle\int_a^\infty f(x)\,dx$ is convergent, then $\displaystyle\int_a^\infty g(x)\,dx$ is convergent.



(b) If $\displaystyle\int_a^\infty g(x)\,dx$ is divergent, then $\displaystyle\int_a^\infty f(x)\,dx$ is divergent.

Ex. Is $\int_0^\infty e^{-x^2}\,dx$ convergent or divergent?



Compare to $g(x) = e^{-x}$

$\boxed{a=1}$

There is some value a s.t. $g(a) = f(a)$

and for $x \geq a$, $0 \leq f(x) \leq g(x)$.

Know $\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx$

$$= \lim_{t\to\infty} -e^{-x}\Big]_{x=1}^{x=t}$$

$$= \lim_{t\to\infty} \left(e^{-1} - e^{-t}\right) = e^{-1}.$$

Comparison thm. $\Rightarrow \int_1^\infty e^{-x^2}\,dx$ is <u>convergent</u>.

$\Rightarrow \int_0^\infty e^{-x^2}\,dx$ is <u>convergent</u>.

Ex. What about $\int_1^\infty \frac{\overbrace{1+e^{-x}}^{f(x)}}{x}\,dx$?

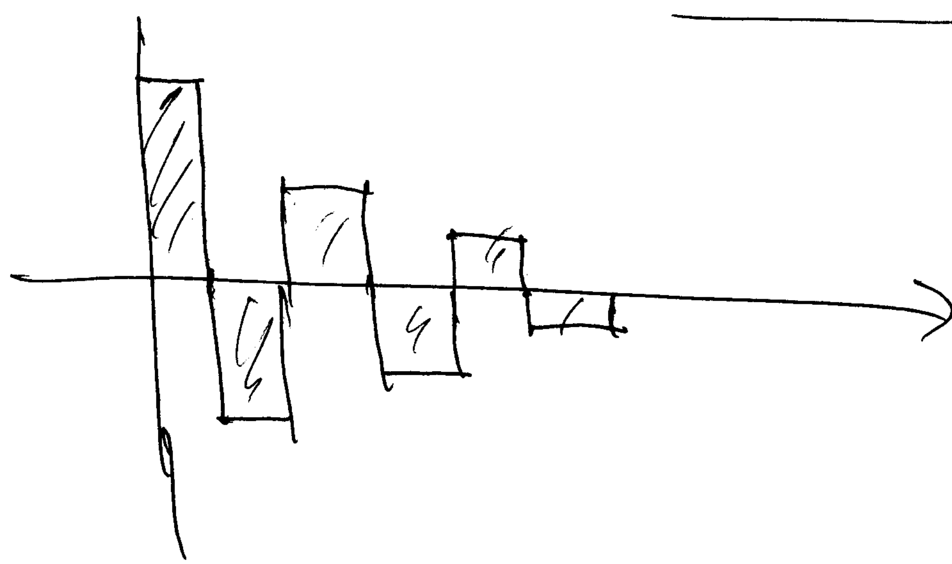
Compare with $g(x) = \frac{1}{x}$.

$\rightsquigarrow$ $f(x) \geq g(x) \geq 0$.

$\rightarrow$ $\int_1^\infty g(x)\,dx = \int_1^\infty \frac{1}{x}\,dx$ is divergent.

$\Longrightarrow$ $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent

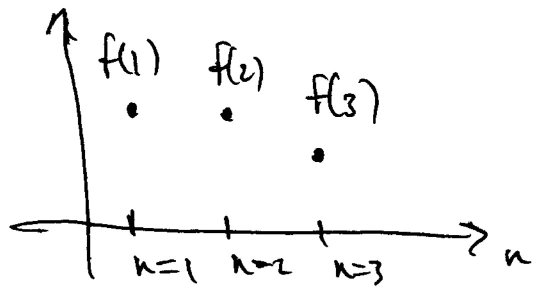
by comparison thm.



$f(x) : \mathbb{R} \to \mathbb{R}$

$\lim_{x \to \infty} f(x) = ?$

$\int_0^\infty f(x)\,dx$



$f(n) : \mathbb{N} \to \mathbb{R}$

$\lim_{n \to \infty} f(n) = ?$

$\sum_{n=1}^{\infty} f(n) = ?$

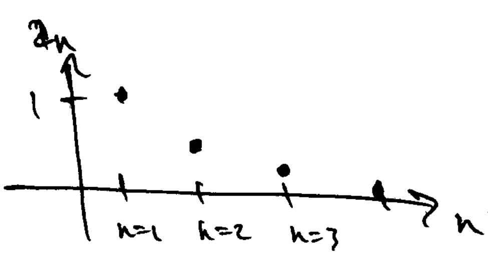
Sequences (12.1)

Let $\{a_1, a_2, a_3, \dots, a_n, \dots\}$ be a list of numbers. We call this a sequence and denote it by $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$.

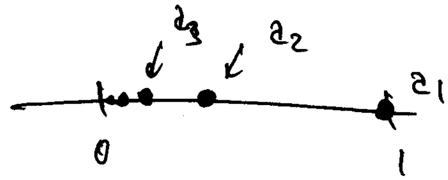
a_n is the n-th term in sequence.

Ex. $\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\}$
$\;\;\uparrow\; a_1 \quad a_2$

$$\Rightarrow a_n = \frac{1}{2n-1}, \quad n \geq 1 \qquad \left(\text{or } \left\{\frac{1}{2n-1}\right\}_{n=1}^{\infty}\right)$$



$\longleftrightarrow$



Ex. $\{1, 1, 2, 3, 5, 8, \dots\}$

"Fibbonaci" sequence.

$$\begin{cases} a_n = a_{n-1} + a_{n-2}, & n \geq 3 \\ a_1 = 1, \ a_2 = 1 \end{cases}$$

Simplest possible expression.

Important question: What is behavior of a_n as $n \to \infty$?

Def. (Limit of sequence)

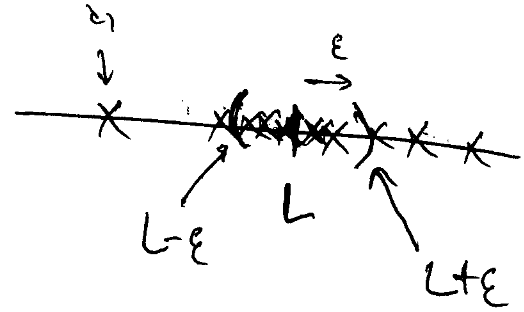
finite!

A sequence $\{a_n\}$ has limit L if we can make a_n as close to L as we like by taking n large enough. In this case, we say that $\{a_n\}$ is convergent and write

$$\lim_{n \to \infty} a_n = L \quad (a_n \to L \text{ as } n \to \infty)$$

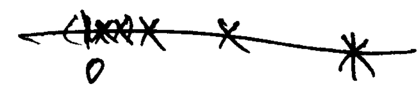
Otherwise, $\{a_n\}$ is divergent.

More precisely, given any $\varepsilon > 0$ (small), after some "time" N (which depends on ε), all the terms $\{a_n, n \geq N\}$ are trapped in $(L-\varepsilon, L+\varepsilon)$.



<u>Ex.</u> Is $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ convergent?

$\lim_{n\to\infty} \frac{1}{n} = 0$ <u>convergent</u>.



<u>Ex.</u> Is $\{1, \underbrace{0}_{1 \text{ zero}}, 1, \underbrace{0, 0}_{2 \text{ zeros}}, 1, \underbrace{0, 0, 0}_{3 \text{ zeros}}, 1, \dots\}$

No, <u>divergent</u> since no matter how large N, there are terms a_n, $n \geq N$ which are equal to 1. So, <u>not</u> trapped in small interval $(-\varepsilon, \varepsilon) \Rightarrow$ divergent.

Ex. Is $\{(-1)^n\}$ convergent?

No, divergent since there is no L s.t. $\lim\limits_{n\to\infty} (-1)^n = L$.

Ex. For what r is $\{r^n\}$ convergent?

$r = -1 \Rightarrow$ divergent

$r = 1 \Rightarrow$ convergent

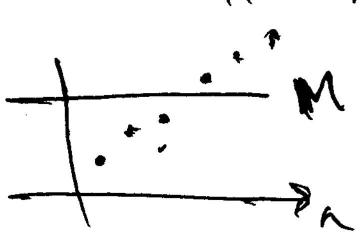
$r = 0 \Rightarrow$ convergent

$$\begin{cases} -1 < r \leq 1 \Rightarrow \text{convergent} \\ r > 1 \text{ or } \leq -1 \Rightarrow \text{divergent.} \end{cases}$$

$r = 2 \Rightarrow \{2^n\}$ divergent.

Def. We say that $\{a_n\}$ diverges to ∞ and write $\lim\limits_{n\to\infty} a_n = +\infty$ if for every $M > 0$, there's some "time" N such that for all $n \geq N$, we have $a_n > M$

large



Limit laws (p. 714)

If $\{a_n\}$, $\{b_n\}$ convergent, and

$a_n \to a$, $b_n \to b$ as $n \to \infty$, then:

1) $a_n + b_n \longrightarrow a+b$ as $n \to \infty$

2) $a_n - b_n \longrightarrow a-b$

3) $a_n b_n \longrightarrow ab$

4) $\dfrac{a_n}{b_n} \longrightarrow \dfrac{a}{b}$ if $b \neq 0$.

5) $a_n^{\,p} \longrightarrow a^p$ if $p>0$ and $a_n>0$

6) $c\,a_n \longrightarrow c\,a$

Thm. If $a_n \to L$ as $n \to \infty$, and f continuous function at L

$$\Rightarrow \quad f(a_n) \to f(L)$$

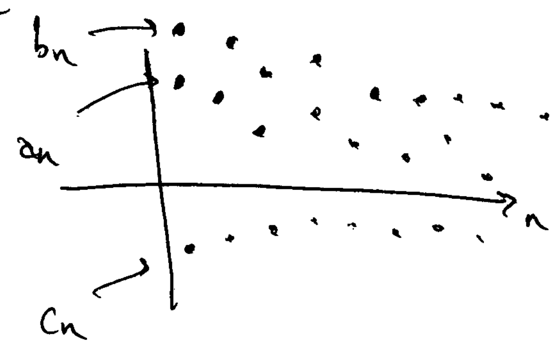
Ex. $\lim\limits_{n\to\infty} \tan\left(\frac{\pi}{n}\right) = ?$

First thing to note is $\lim\limits_{n\to\infty} \frac{\pi}{n} = 0$.

And, $f(x) = \tan(x)$ is continuous at $x = 0$.

$$\Rightarrow \lim_{n\to\infty} \tan\left(\frac{\pi}{n}\right) = \tan\left(\lim_{n\to\infty} \frac{\pi}{n}\right) = \tan 0 = 0.$$

Other useful tools for convergence:



Squeeze theorem:

If $c_n \le a_n \le b_n$ and

$\lim_{n\to\infty} c_n = \lim_{n\to\infty} b_n = L$,

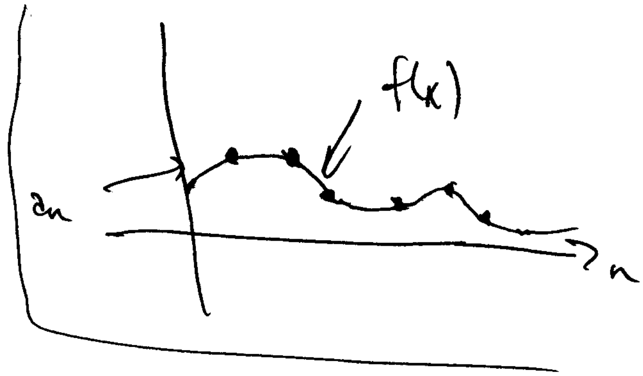
then $\lim_{n\to\infty} a_n = L$.

In particular, if $|a_n| \to 0$ then

$a_n \to 0$ as $n \to \infty$.

(since $-|a_n| \le a_n \le |a_n|$.)

Interpolation theorem:



Suppose there is a function $f(x)$ s.t. $f(n) = a_n$ for all integers n. Then if $\lim_{x \to \infty} f(x) = L$, we have that

$$\lim_{n \to \infty} a_n = L.$$

($\Leftarrow$ not true!)

Monotonicity + boundedness:

Def. 1) $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$.

2) $\{a_n\}$ decreasing $a_n > a_{n+1}$ for all $n \geq 1$.

$\{a_n\}$ monotonic if it is either increasing or decreasing.

Ex. $\{(-1)^n\}$ not monotonic!

$\{\frac{1}{n}\}$ is monotonic!

i.e., monotonicity $\Rightarrow$ no oscillations!

Def. 1) $\{a_n\}$ is bounded above if there is M s.t. $a_n \leq M$ for all $n \geq 1$.

2) bounded below

m s.t. $a_n \geq m$ for all $n \geq 1$

$\{a_n\}$ bounded if it is both bounded above and bounded below.

Ex. $\{2^n\}$ is not bounded!

$\{\frac{1}{n}\}$ is bounded!

i.e., bounded $\Rightarrow$ no going to ∞!

Thm. If $\{a_n\}$ bounded and monotonic, then $\{a_n\}$ is convergent.

Series (12.2)

$\{a_n\}_{n=1}^{\infty}$ sequence.

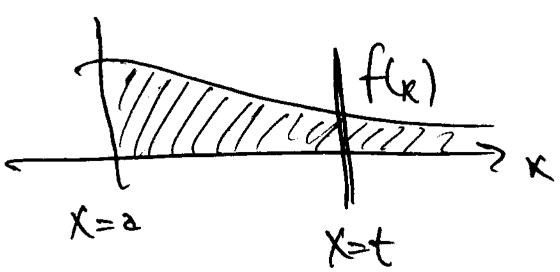
Does it make sense to consider $a_1 + a_2 + \dots + a_n + \dots$

We call this an <u>(infinite) series</u>,

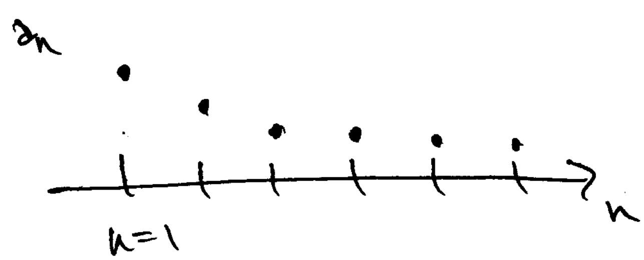
denoted $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$.

How can we add an infinite no. of terms?

Recall idea of improper integrals:



$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$



$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \underline{\sum_{i=1}^{n} a_i}$$

dummy variable

<u>Def.</u> Define the <u>partial sum</u>

$$s_n = a_1 + \dots + a_n = \sum_{i=1}^{n} a_i, \quad n \geq 1.$$

If $\{s_n\}$ <u>convergent</u> and $\lim_{n \to \infty} s_n = s$ exists as a real number, we say that $\sum_{i=1}^{\infty} a_i$ is <u>convergent</u> and define

$$\sum_{i=1}^{\infty} a_i = s.$$

"sum of the series"

Otherwise, the series is <u>divergent</u>.

<u>Ex.</u> Is $\sum_{n=1}^{\infty} (-1)^n$ convergent?

$$s_n = \sum_{i=1}^{n} (-1)^i = \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

s_n is <u>not convergent</u>! $\Rightarrow$ $\sum_{n=1}^{\infty} (-1)^n$ <u>divergent</u>!

<u>Ex.</u> (Harmonic series)

Is $\sum_{n=1}^{\infty} \frac{1}{n}$ convergent? $\quad \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$$1 = s_1 = 1$$

$$1.5 \le s_2 = 1 + \frac{1}{2}$$

$$2 \le s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$

$$> \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

$$2.5 \le s_8 = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}}$$

i.e., <u>diverges</u> to $+\infty$!

<u>Ex.</u> $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent ... next time