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Functions of multiple variables (15.1)

Now we discuss functions of two variables $f(x,y)$ which assign a real number z to each pair $(x,y) \in \mathbb{R}^2$ $(z = f(\underbrace{x,y}_{\text{indep. variables}}))$.

dependent variable

Actually, consider $(x,y) \in D \subset \mathbb{R}^2$

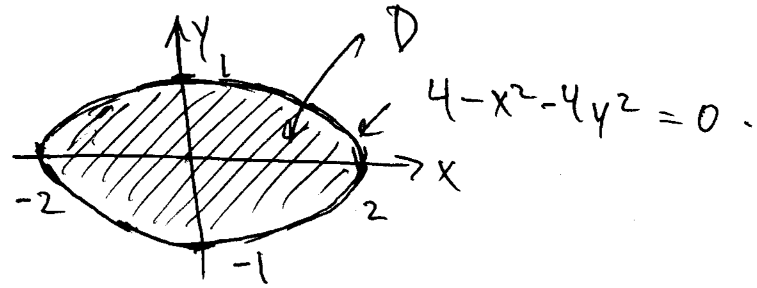
D is the domain of f and

$R = \{ f(x,y) : (x,y) \in D \}$ is the range of f.

Unless we specify D, we take the domain to be all (x,y) s.t. the function $f(x,y)$ is well-defined.

Ex. Domain, range of $f(x,y) = \sqrt{4 - x^2 - 4y^2}$?

$D =$ all $(x,y) \in \mathbb{R}^2$ s.t. $4 - x^2 - 4y^2 \geq 0$

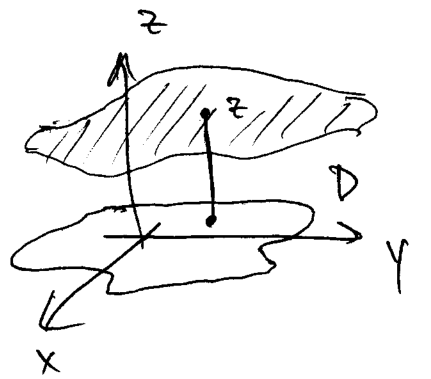


$R = [0, 2]$.

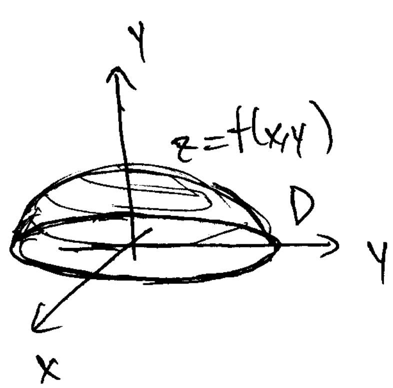
Graphing functions of two variables

The **graph** of $f(x,y)$ is the set of all $(x,y,z) \in \mathbb{R}^3$ s.t. $z = f(x,y)$ and $(x,y) \in D$.

For example, think of (x,y) as being the coordinates of a location on a terrain map and $z = f(x,y)$ as the elevation.



Ex. $z = f(x,y) = \sqrt{4 - x^2 - 4y^2}$



$z^2 = 4 - x^2 - 4y^2$ with $z \geq 0$

$\Rightarrow \quad x^2 + 4y^2 + z^2 = 4, \quad z \geq 0$

graph of top half of an ellipsoid!

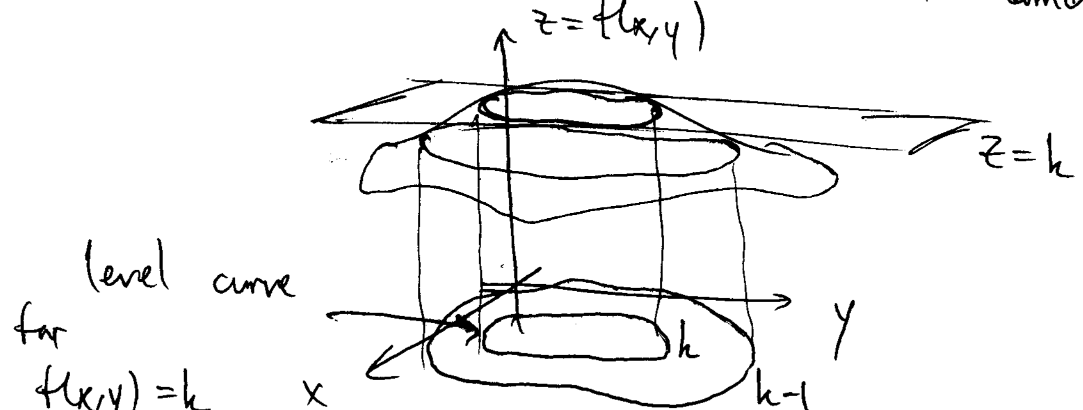
<u>Ex.</u> $z = f(x,y) = \underbrace{ax + by + c}_{\text{linear function}}$, a, b, c constants

$z = ax + by + c$ plane with normal vector

$ax + by - z + c = 0$ $\qquad \vec{n} = (a, b, -1)$.

<u>Level curves</u>

Instead of drawing the graph of the surface $z = f(x,y)$ in 3-D, we visualize the function by drawing a contour map in 2-D. Do this by drawing <u>level curves</u> of f, which are curves in x-y plane (restricted to domain) s.t. $f(x,y) = $ constant.



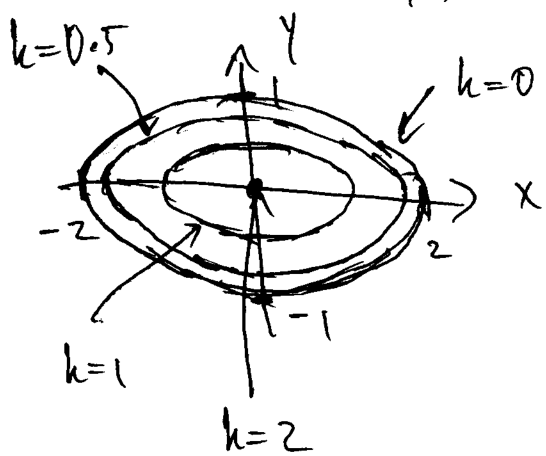
Note:

1) Usually choose values k for constant in some equally spaced grid, e.g., $k \in \{-5, 0, 5, 10, 15, 20\}$.
$f(x,y) = -5$, $f(x,y) = 0$, ...

2) Level curves of "smooth" functions are closed, do not cross each other.

3) Where level curves are bunched closer together the function changes more rapidly.

Ex. $f(x,y) = \sqrt{4 - x^2 - 4y^2}$



0.5, 1.5
Let's choose $k \in \{0, 1, 2\}$ and find level curves $f(x,y) = k$.

Functions of more than two variables

A function f of n variables takes n-tuples $(x_1, \dots, x_n) \in \mathbb{R}^n$ to $\mathbb{R}$:

$$z = f(x_1, \dots, x_n).$$

We can study such functions by considering their level surfaces:

$$f(x_1, \dots, x_n) = k \quad \text{for some constant } k.$$

Ex. $f(x_1, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$

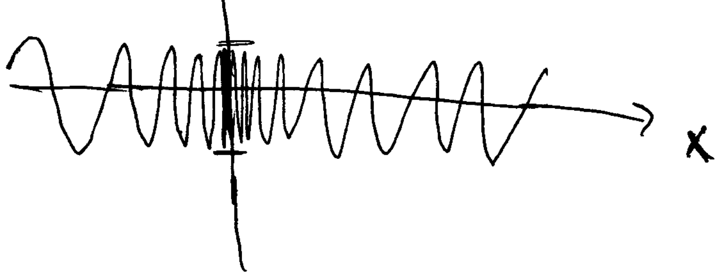
linear function of n variables.

Can write this as $f(\vec{x}) = \vec{c} \cdot \vec{x}$

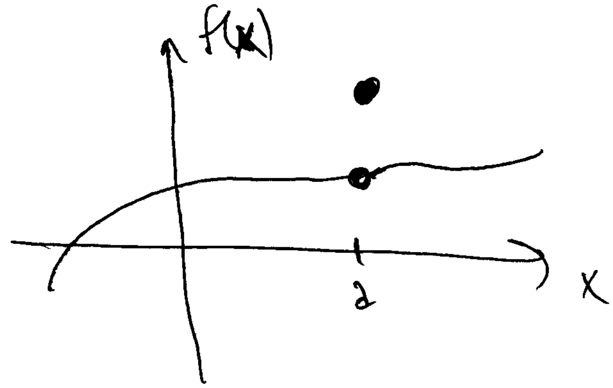
with $\vec{x} = (x_1, \dots, x_n)$

$\vec{c} = (c_1, \dots, c_n)$

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$



No limit exists at $x = a$



Limit L exists at $x = a$ but $f(a) \neq L$.

<u>For functions $f(x,y)$ of two variables:</u>

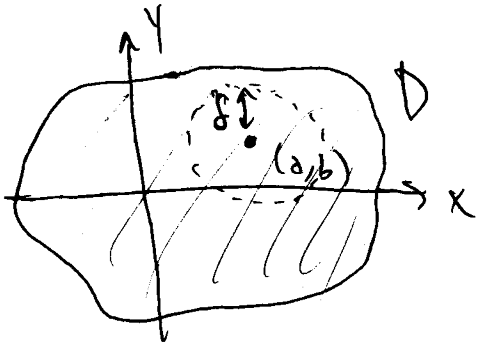
<u>Def.</u> Suppose $(a,b) \in D$ (domain) of function f. Then <u>the limit of $f(x,y)$ at (a,b) is L</u> $\left(\lim\limits_{(x,y) \to (a,b)} f(x,y) = L \right)$ if

for every $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ s.t. for all $(x,y) \in D$ with

$$0 < \underbrace{\sqrt{(x-a)^2 + (y-b)^2}}_{} < \delta$$

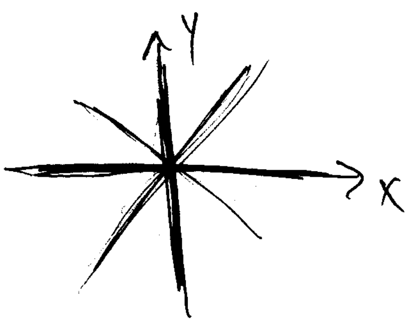
we have that ← distance between (x,y) and (a,b).

$$|f(x,y) - L| < \varepsilon.$$

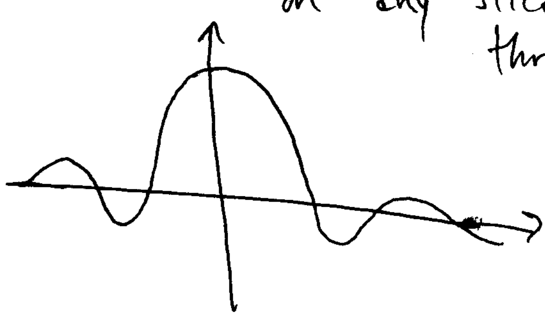


Ex. Limit as $(x,y) \to (0,0)$ of

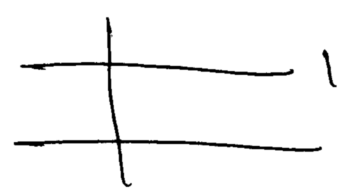
$$f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2} \quad \text{vs.} \quad f(x,y) = \frac{x^2-y^2}{x^2+y^2}$$



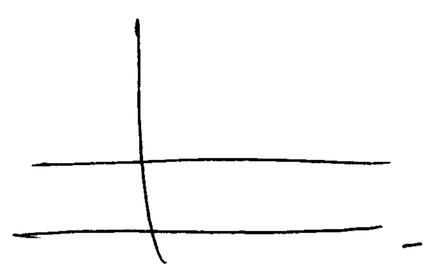
on any slice through origin



on slice through x-axis, looks like

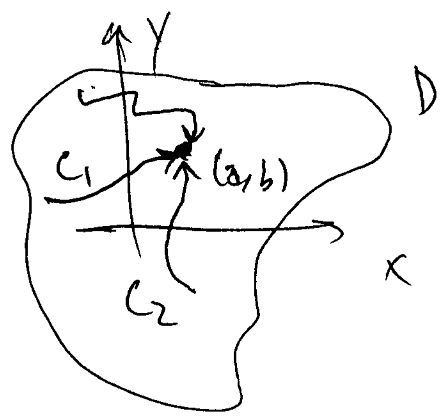


on slice through y-axis, looks like



One way to test that $f(x,y)$ <u>doesn't</u> have a limit as $(x,y) \to (a,b)$:

If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along path C_1, and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along path C_2, and $L_1 \neq L_2$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ does <u>not</u> exist.



<u>Ex.</u> $f(x,y) = \dfrac{-xy^2}{x^2+y^4}$ $\qquad y = mx$

limit as $(x,y) \to (0,0)$?

As you approach $(0,0)$ along $x = -y^2$,

$f(x,y) \to \frac{1}{2} \neq 0$.

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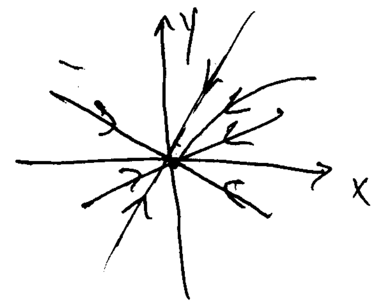
Ex. $f(x,y) = \dfrac{-xy^2}{x^2+y^4}$

limit as $(x,y) \to (0,0)$



Let's test along paths $y = mx$, m constant.

$$\Rightarrow \quad f(x, y(x)) = \frac{-m^2 x^3}{x^2 + m^4 x^4} \xrightarrow[x \to 0]{} 0$$

Let's test along path $x = y^2$:

$$\Rightarrow \quad f(x(y), y) = \frac{-y^4}{2y^4} = -\frac{1}{2} \xrightarrow[y \to 0]{} -\frac{1}{2}$$

$\Rightarrow$ limit doesn't exist!

Remark: Can't simply test along straight paths! Must treat on case-by-case basis.

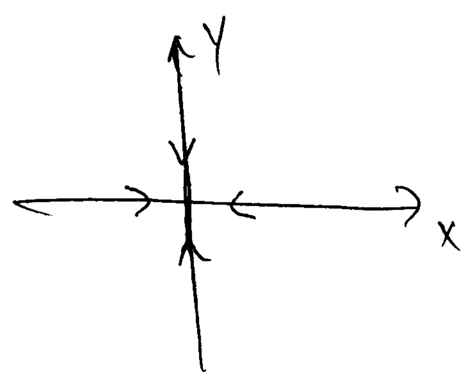
Note: Limit laws that you've learned for functions of a single variable also hold for functions of multiple variables!

(e.g., $\lim_{(x,y)\to(a,b)} \left(f(x,y)\, g(x,y) \right) =$

$$\left(\lim_{(x,y)\to(a,b)} f(x,y) \right) \left(\lim_{(x,y)\to(a,b)} g(x,y) \right)$$

if each limit exists.)

Ex. $f(x,y) = \dfrac{4x^2 y}{x^2+y^2}$



limit as $(x,y) \to (0,0)$

Test along $x=0$:

$f(x,y) = 0 \longrightarrow 0$

Test along $y=0$:

$f(x,y) = 0 \longrightarrow 0$.

Test along path which balances the terms in the denominator, in this case $x(y) = y$:

$$f(x(y),y) = \frac{4y^3}{2y^2} = 2y \xrightarrow[y\to 0]{} 0$$

Let's guess that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

<u>Pf.</u>

$$0 \le |f(x,y) - 0| = \left|\frac{4x^2y}{x^2+y^2}\right| = \frac{4x^2|y|}{x^2+y^2}$$

$$\le \frac{4\cancel{(x^2+y^2)}|y|}{\cancel{x^2+y^2}}$$

$$= 4|y| \xrightarrow[y\to 0]{} 0$$

$f(x,y) \to 0$ as $(x,y) \to (0,0)$.

<u>Continuity</u>

<u>Def.</u> $f(x,y)$ is <u>continuous</u> at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

f is <u>continuous</u> if it is continuous at every $(a,b) \in D$.

<u>Note</u>: All <u>polynomials</u> are continuous

$$f(x,y) = \sum_{m=0}^{M} \sum_{n=0}^{N} c_{nm} \, x^m y^n .$$

$\uparrow$ constants

are continuous functions on $\mathbb{R}^2$.

For ex., $f(x,y) = 7x^2y - 3xy^3 + 17x - 7$ is continuous on $\mathbb{R}^2$.

<u>Note</u>: Sums, differences, products and quotients of continuous functions are continuous <u>on their domains</u>.

(e.g. $f(x,y) = \dfrac{g(x,y)}{h(x,y)}$ for g, h cont.

is also cont. on $D = \{ (x,y) : h(x,y) \neq 0 \}$.

In addition, $f(x,y) = F(g(x,y))$

(write $f = F \circ g$)

is continuous if g and F are.

Ex. Let $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = 0 \end{cases}$

1) Continuous at $(0,0)$?

Since $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

does not exist,

function is not cont. at $(0,0)$.

2) Where is $f(x,y)$ continuous?

Cont. everywhere except at $(0,0)$.

Ex. $f(x,y) = \begin{cases} \dfrac{4x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Where is f continuous?

Here, f is continuous everywhere, in particular, is cont. at $(0,0)$ since

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0).$$

Functions of more than two variables

Def. Let $\vec{x} = (x_1, \dots, x_n)$, $\vec{a} = (a_1, \dots, a_n)$

We say that the limit of the function $f(\vec{x})$ at $\vec{x} = \vec{a}$ is L (write this as $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = L$) if for every $\varepsilon > 0$ there is a $\delta > 0$ s.t. if $|\vec{x} - \vec{a}| < \delta$ then

$$|f(\vec{x}) - L| < \varepsilon.$$

Def. $f(\vec{x})$ <u>continuous</u> at $\vec{a}$ if

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{a}.$$

Partial derivatives (13.3)

For functions $f(x)$ of one variable,

$\frac{df}{dx}$ measured how much f changed

if x changed (rate of change):

$$\delta f = f(x+\delta x) - f(x)$$

$$\frac{df}{dx} \approx \frac{\delta f}{\delta x} \quad \text{as} \quad \delta x \to 0$$

For functions $f(x,y)$ of two variables,

we do something similar:

1) To see how much f changes if x changes (while keeping y fixed):

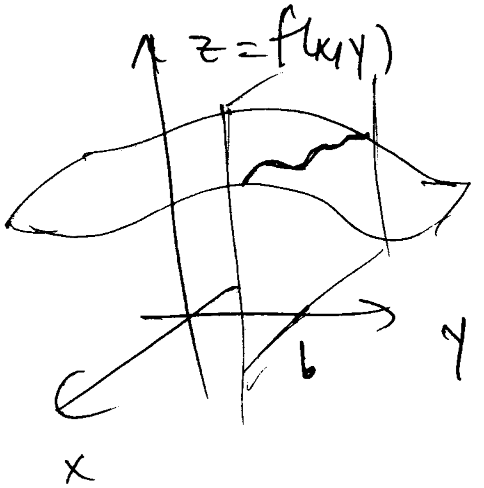
$$\delta f = f(x+\delta x, y) - f(x,y)$$

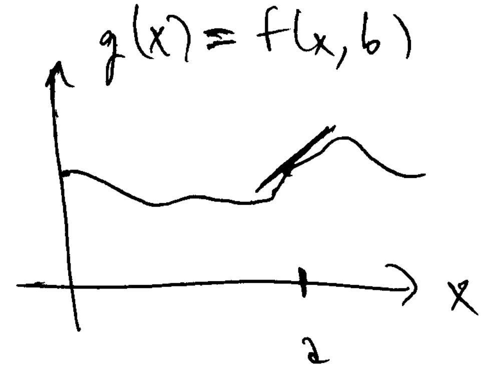
$$f_x(x,y) = \frac{\delta f}{\delta x} \quad \text{as} \quad \delta x \to 0.$$

i.e, the <u>partial derivative of f with respect to x</u> at (a,b) is

$$\boxed{f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}}$$

if the limit exists.





We think of $f_x(a,b) = g'(a)$

where $g(x) = f(x,b)$

2) Similarly, the partial derivative of f with respect to y at (a,b) is

$$\boxed{f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}}$$

We think of

$f_y(a,b) = h'(b)$ $h(y) = f(a,y)$

Notation: We will often write the partial derivatives of $f(x,y)$ w.r.t. x or y as

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x} f(x,y) = D_x f(x,y)$$

derivative w.r.t. x

arguments of f_x

$$f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y} f(x,y) = D_y f(x,y).$$

Remark: When taking partial derivatives of f w.r.t. x, treat y (and ~~other~~ any other variables) as a constant.

Ex. $f(x,y) = 3xy^4 + x^3y^2 - \sin(xy)$

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, evaluate at $(x,y) = (3,2)$

i) $\dfrac{\partial f}{\partial x} = 3y^4 + 3x^2y^2 - y\cos(xy)$

$$\frac{\partial f}{\partial x}(3,2) = 3\cdot 16 + 12\cdot 9 - 2\cos(6)$$
$$= 156 - 2\cos(6).$$

ii) $\dfrac{\partial f}{\partial y} = 12xy^3 + 2x^3y - x\cos(xy)$

$$\frac{\partial f}{\partial x}(3,2) = 396 - 3\cos(6).$$

<u>Ex.</u> (Implicit differentiation)

What is $\dfrac{\partial z}{\partial x}$ if $z = z(x,y)$ is defined implicitly by

$$x^3 + y^3 + z^3 + 6xyz = 1$$

$$\frac{\partial}{\partial x}\left(x^3 + y^3 + z(x,y)^3 + 6xy\,z(x,y)\right) = 0$$

$$= 3x^2 + 0 + 3z(x,y)^2\frac{\partial z}{\partial x}$$
$$+ 6y\,z(x,y) + 6xy\frac{\partial z}{\partial x}$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)\left(3z^2+6xy\right) = -3x^2-6yz$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{-3x^2-6yz}{3z^2+6xy}.}$$

Higher derivatives.

$\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are still functions of two variables. So we consider partial derivatives of these functions if the limit exists:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

<u>Remark</u>: The mixed partials f_{xy} and f_{yx} are not necessarily the same!

<u>Thm.</u> (Clairaut's thm.) If f is defined on domain D and f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$ on D.

(Also true for higher derivatives:

$f_{xxy} = f_{xyx} = f_{yxx}$ if these are all continuous.

Ex. $f(x,y) = 7x^3 - 3x^2y^2 + x^2y - 3y^2 + y^5$

$f_x(x,y) = 21x^2 - 6xy^2 + 2xy.$

$f_y(x,y) = -6x^2y + x^2 - 6y + 5y^4$

$f_{xy}(x,y) = -12xy + 2x$

$f_{yx}(x,y) = -12xy + 2x$

$f_{xx}(x,y) = 42x - 6y^2 + 2y$

$f_{xxy}(x,y) = -12y + 2$

$f_{xyx}(x,y) = -12y + 2$

$f_{yxx}(x,y) = -12y + 2$