Functions of more than two variables

Suppose $\vec{x} = (x_1, \ldots, x_n)$ and $f(\vec{x}) = f(x_1, \ldots, x_n)$

Def. The partial derivative of f with respect to x_i is

$$f_{x_i}(\vec{x}) = \frac{\partial f}{\partial x_i}(\vec{x}) = \lim_{h \to 0} \left[\frac{\left(f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)\right)}{h} \right]$$

when limit exists.

Remark: To find $\frac{\partial f}{\partial x_i}$, treat all x_j, $j \neq i$ as constants then take derivative w.r.t. x_i.

Ex. If $f(x,y,z) = z(\ln y)e^{xz}$, find f_{xyz}.

$$f_x = \frac{\partial f}{\partial x} = z^2(\ln y)e^{xz}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = z^2 \cdot \frac{1}{y} \cdot e^{xz}$$

$$f_{xyz} = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = 2z \cdot \frac{1}{y} e^{xz} + z^2 \cdot \frac{1}{y} \cdot x e^{xz}$$

$$= e^{xz}\left(\frac{2z}{y} + \frac{z^2x}{y}\right).$$

Partial differential equations (PDE)

Used to model phenomena described by rates of change.

Ex. (Wave eqn.)

Let $u = u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$ be the height u of a string at position x at time t.

The function $u(x,t)$ satisfies the PDE

$$\star \quad \boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

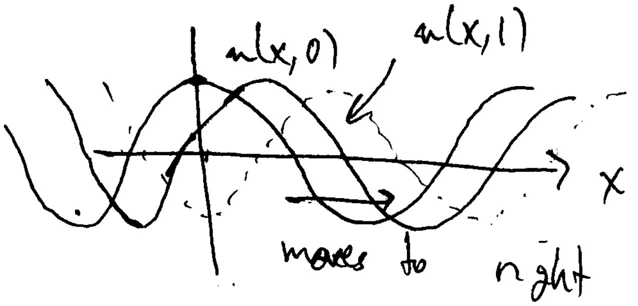
with a given constant c.

Verify that $u(x,t) = \cos(x - ct)$ is a solution of the PDE $\star$:

$u_x = -\sin(x-ct)$ $\qquad$ $u_t = -\sin(x-ct)\cdot(-c)$

$u_{xx} = -\cos(x-ct)$ $\qquad$ $u_{tt} = -\cos(x-ct)\cdot(-c)^2$

$\Rightarrow u_{tt} = c^2 u_{xx}$. ✓



Can also check that

$u(x,t) = \cos(x+ct)$ is also a solution to the PDE ☆ (wave moving to left at speed c).

Later on, we'll show that any function of the form

$$u(x,t) = f(x-ct) + g(x+ct)$$

is a soln. to ☆, f and g are arbitrary twice differentiable functions.

<u>Remark</u>: In general, if given a PDE in n-variables $x_1, \dots, x_n$, we should at least be able to <u>check</u> if a given function $u(x_1, \dots, x_n)$ solves it.

Ex. Suppose we have the system of PDE

Douglas - Cobb PDE
$$\begin{cases} \dfrac{\partial P}{\partial L} = \alpha \dfrac{P}{L} \\ \dfrac{\partial P}{\partial K} = \beta \dfrac{P}{K} \end{cases}, \quad \text{with } \alpha, \beta > 0 \text{ constants.}$$

P = production, L = labor, K = capital

We seek a soln. $P = P(L, K)$

Check that $\boxed{P(L,K) = c L^{\alpha} K^{\beta}}$

w/ constant c is a soln. to PDE:

$$\frac{\partial P}{\partial L} = c\alpha L^{\alpha-1} K^{\beta} = \alpha \frac{cL^{\alpha}K^{\beta}}{L}$$

$$= \alpha \frac{P}{L}. \quad \checkmark$$

$$\frac{\partial P}{\partial K} = c\beta L^{\alpha} K^{\beta-1} = \beta \frac{cL^{\alpha}K^{\beta}}{K}$$

$$= \beta \frac{P}{K}. \quad \checkmark$$

Ex. (Black - Scholes)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

V = value of option $\qquad r, \sigma$ constants.

t = time

S = stock price

Seek soln's $V = V(S,t)$

→ Has to be done numerically.

<u>Chain rule (15.5)</u>

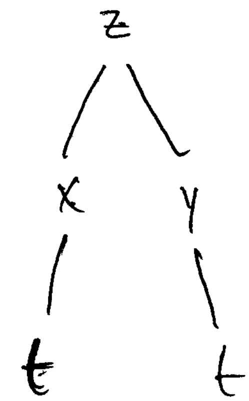
Recall that if $y = f(x)$ and $x = g(t)$ then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \qquad \left((f(g(t)))' = f'(g(t))g'(t) \right)$$

Now suppose $z = f(x,y)$ is a differentiable function of x,y and $x = g(t)$ and $y = h(t)$ are differentiable functions of t.

What is $\frac{dz}{dt}$?

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$



Ex. $z = x^2 + y^2 + xy$, $x = \sin t$, $y = e^t$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

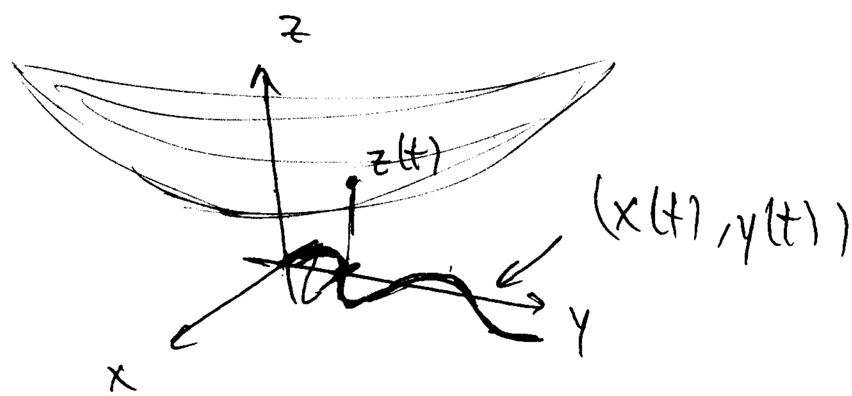
$$= (2x+y)(\cos t) + (2y+x)(e^t)$$

$$= (2\sin t + e^t)(\cos t) + (2e^t + \sin t)e^t.$$

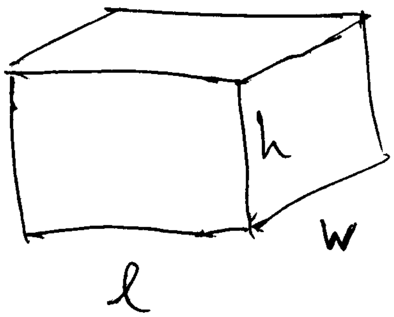
What is $\frac{dz}{dt}$ at $t=0$? Here, $\left.\frac{dz}{dt}\right|_{t=0} = 3$.

Interpretation:

How does z change as we follow parametric curve $(x(t), y(t))$?



Ex.



Box dimensions change with time t.

At time $t=0$, $\ell = 1m$, $w = h = 2m$.
and ℓ and w increasing with rate $2m/s$ while h decreases with rate $3m/s$.

a) Rate of change of volume of box at $t=0$?

$V = V(\ell, w, h) = \ell w h$.

What is $\left.\frac{dV}{dt}\right|_{t=0}$?

$$\frac{dV}{dt} = \underbrace{\frac{\partial V}{\partial \ell}}_{wh}\frac{d\ell}{dt} + \underbrace{\frac{\partial V}{\partial w}}_{\ell h}\frac{dw}{dt} + \underbrace{\frac{\partial V}{\partial h}}_{\ell w}\frac{dh}{dt}$$

$$\left.\frac{dV}{dt}\right|_{t=0} = \left.\left(wh\frac{d\ell}{dt}\right)\right|_{t=0} + \left.\left(\ell h\frac{dw}{dt}\right)\right|_{t=0} + \left.\left(\ell w\frac{dh}{dt}\right)\right|_{t=0}$$

$$= 2\cdot 2\cdot 2 + 1\cdot 2\cdot 2 + 1\cdot 2\cdot(-3)$$

$$= 6 \ m^3/s.$$

b) Rate of change of surface area at $t=0$?

$$S = S(l,w,h) = 2(lw + lh + wh)$$

$$\left.\frac{dS}{dt}\right|_{t=0} \;?$$

c) Rate of change of length of diagonal at $t=0$?

$$D = D(l,w,h) = \sqrt{l^2 + w^2 + h^2}$$

$$\left.\frac{dD}{dt}\right|_{t=0} = \;?$$

More general version of chain rule:

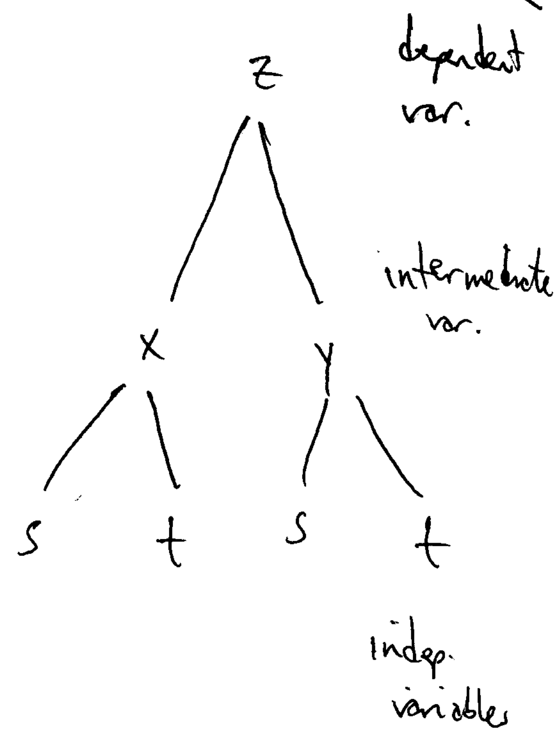
Now suppose $z = f(x,y)$, $x = g(s,t)$ and $y = h(s,t)$

Then we can consider z as a function of s,t.

How to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$?

Then,
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



<u>Note:</u> To find $\frac{dz}{ds}$, keep track of all paths in tree diagram that go from z to s.

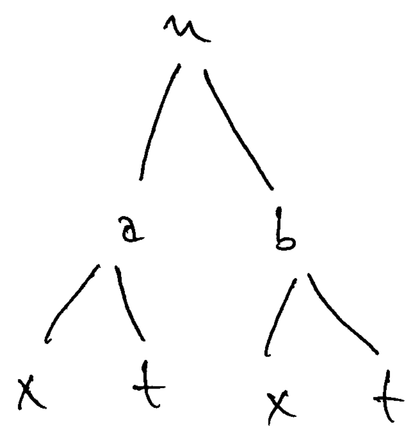
<u>Chain rule (most general):</u>

If $u = u(x_1, \dots, x_n)$ and $x_j = x_j(t_1, \dots, t_m)$ then for each $i = 1, \dots, m$,

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt_i} + \dots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt_i}}$$

<u>Ex.</u> Show that $u(x,t) = f(x-ct) + g(x+ct)$ solves the PDE $u_{tt} = c^2 u_{xx}$, for any twice differentiable functions f, g.

Let $a = x - ct$, $b = x + ct$, so $u = f(a) + g(b)$



Then,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a}\frac{da}{dx} + \frac{\partial u}{\partial b}\frac{db}{dx}$$

$$= f'(a)\cdot 1 + g'(b)\cdot 1$$

$$= f'(a) + g'(b).$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \underbrace{\frac{\partial}{\partial a}\left(\frac{\partial u}{\partial x}\right)}\frac{da}{dx} + \underbrace{\frac{\partial}{\partial b}\left(\frac{\partial u}{\partial x}\right)}\frac{db}{dx}$$

$$= f''(a)\cdot 1 + g''(b)\cdot 1$$

$$= f''(a) + g''(b).$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial a}\frac{da}{dt} + \frac{\partial u}{\partial b}\frac{db}{dt}$$

$$= f'(a)\cdot(-c) + g'(b)\cdot(+c)$$

$$= -c\left(f'(a) - g'(b)\right).$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)$$

$$= \underbrace{\frac{\partial}{\partial a}\left(\frac{\partial u}{\partial t}\right)}\frac{da}{dt} + \underbrace{\frac{\partial}{\partial b}\left(\frac{\partial u}{\partial t}\right)}\frac{db}{dt}$$

$$= \left(-c f''(a)\right)\cdot(-c) + \left(+c g''(b)\right)\cdot(+c)$$

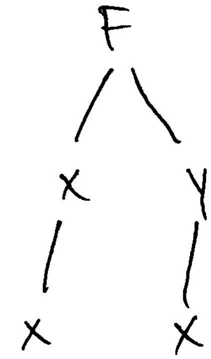
$$= c^2\left(f''(a) + g''(b)\right).$$

$$= c^2\,\frac{\partial^2 u}{\partial x^2}. \quad \checkmark$$

Implicit differentiation

Suppose $F(x, y) = 0$ defines y implicitly as a function of x, i.e, $y = f(x)$.

What is $\frac{dy}{dx}$?



$$F(x, f(x)) = 0$$

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} \cdot \cancelto{1}{\frac{dx}{dx}} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \quad \text{if } F_y \neq 0.}$$

Ex. $\frac{dy}{dx}$ if $x^4 - \sin(xy) = 6x^2y^2$?

$$F(x,y) = x^4 - \sin(xy) - 6x^2y^2 = 0$$

defines y as a function of x.

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{4x^3 - y\cos(xy) - 12xy^2}{-x\cos(xy) - 12x^2y}.$$

when $-x\cos(xy) - 12x^2y \neq 0$.

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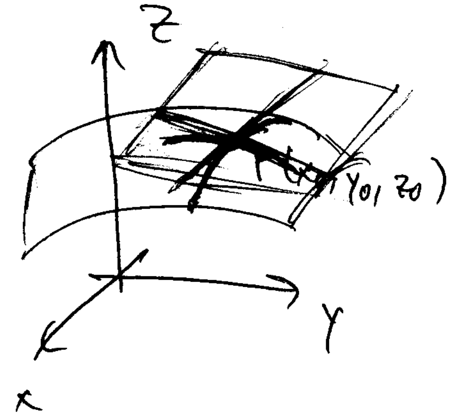
Tangent planes (15.4)

Suppose $z=f(x,y)$ describes a surface.
How to find tangent plane at a point $P(x_0, y_0, z_0)$ on surface?



Planes must be of the form

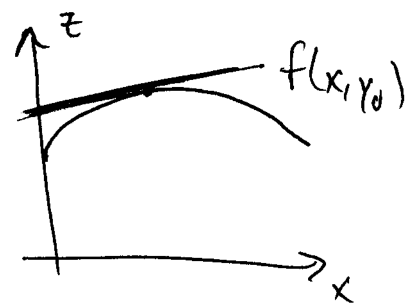
$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$$

$$\Rightarrow \quad z - z_0 = a(x-x_0) + b(y-y_0), \quad a, b \text{ constants}$$

What are a, b?

Trace of surface $z=f(x,y)$ on $y=y_0$ is $z-z_0 = a(x-x_0)$

$$\Rightarrow \quad a = f_x(x_0, y_0)$$

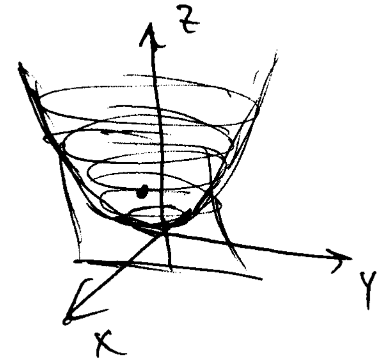


Similarly, $b = f_y(x_0, y_0)$.

$$\Rightarrow \quad \boxed{z - z_0 = f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0)}$$

This is the eqn. for the tangent plane to the surface at $P(x_0, y_0, z_0)$.

<u>Ex.</u> Find tangent plane to $z = x^2 + y^2$ at $(1,1,2)$.



$f(x,y) = x^2 + y^2$

$f_x(x,y) = 2x \Rightarrow f_x(1,1) = 2$

$f_y(x,y) = 2y \Rightarrow f_y(1,1) = 2$

$$\Rightarrow \boxed{z - 2 = 2(x-1) + 2(y-1)}$$

<u>Linear approximation</u>

Rewriting the eqn. for the tangent plane with $L(x,y) = z$, $f(x_0, y_0) = z_0$:

$$\boxed{L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

linear approximation of f at (x_0, y_0)

For (x,y) close to (x_0,y_0),

$L(x,y) \approx f(x,y)$ (L is actually the first-degree

Taylor polynomial of $f(x,y)$ at (x_0, y_0)!)

We say that $f(x,y)$ is differentiable at (a,b) if is "well-approximated" by its linear approximation $L(x,y)$. as $(x,y) \to (a,b)$:

$$f(a+\delta x, b+\delta y) - f(a,b)$$
$$= f_x(a,b)\,\delta x + f_y(a,b)\,\delta y$$
$$+ \varepsilon_1 \delta x + \varepsilon_2 \delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\delta x, \delta y) \to (0,0)$.

Easier to check differentiability using following theorem:

Thm. If f_x and f_y exist near (a,b) and are continuous, then f is differentiable at (a,b).

<u>Ex.</u> Show $f(x,y) = x e^{xy}$ differentiable at $(1,0)$ and find linearization.

$$f_x(x,y) = e^{xy} + xy e^{xy}$$
$$f_y(x,y) = x^2 e^{xy}$$
$\Rightarrow f$ diff. at $(1,0)$

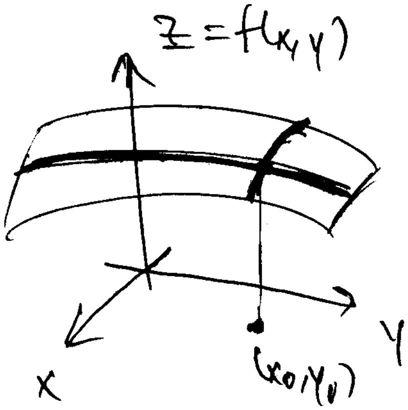
$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$
$$= 1 + 1\cdot(x-1) + 1\cdot(y-0)$$
$$= x + y.$$

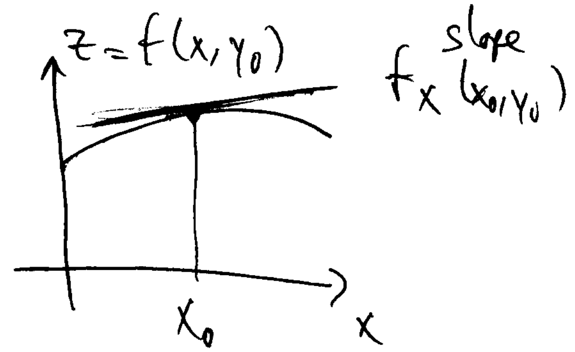
So, $x e^{xy} \approx x+y$ near $(1,0)$.

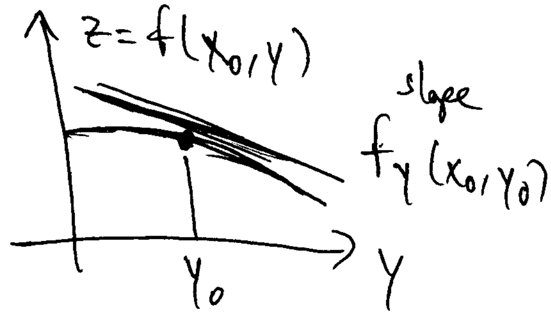
<u>Remark:</u> We can write linear approximation in terms of <u>differentials</u> dx and dy as

$$\boxed{\begin{aligned} dz &= f_x(x,y)\,dx + f_y(x,y)\,dy \\ &= \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy. \end{aligned}}$$

Directional derivatives and gradient vector (15.6)

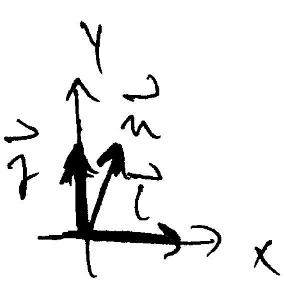






We can think of f_x and f_y as rates of change of z in the x- and y- directions, i.e, in the directions $\underbrace{\vec{i} \text{ and } \vec{j}}_{\text{unit vectors}}$.

What if we take the derivative in some arbitrary direction $\vec{u} = (a, b)$ for some unit vector $\vec{u}$?



With $\vec{x}_0 = (x_0, y_0)$ and $\vec{x} = (x, y)$ and $f(x,y) = f(\vec{x})$, we had that

$$f_x(x,y) = f_x(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{i}) - f(\vec{x}_0)}{h}$$

$$f_y(x,y) = f_y(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{j}) - f(\vec{x}_0)}{h}$$

For any <u>unit</u> vector $\vec{u}$, define the <u>directional derivative</u> of f at $\vec{x}_0 = (x_0, y_0)$ in the direction $\vec{u} = (a,b)$ as

$$D_{(a,b)} f(x_0, y_0) = D_{\vec{u}} f(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

<u>Ex.</u> $D_{\vec{i}} f(\vec{x}_0) = f_x(\vec{x}_0)$

$D_{\vec{j}} f(\vec{x}_0) = f_y(\vec{x}_0)$.

<u>Thm.</u> If f differentiable in x and y, it has a directional derivative for any <u>unit</u> vector $\vec{u} = (a,b)$ and

$$\boxed{D_{\vec{u}} f(\vec{x}_0) = f_x(\vec{x}_0)\, a + f_y(\vec{x}_0)\, b}$$

Pf.

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0 + hb) + f(x_0, y_0 + hb) - f(x_0, y_0)}{h}$$

$$= a \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0 + hb)}{ha}$$

$$+ b \lim_{h \to 0} \frac{f(x_0, y_0 + hb) - f(x_0, y_0)}{hb}$$

$$= a f_x(x_0, y_0) + b f_y(x_0, y_0).$$

Def. The gradient of f is the vector

$$\boxed{\begin{aligned} \vec{\nabla} f(x,y) &= D f(x,y) = \overrightarrow{\text{grad}}\, f \\ &= (f_x(x,y), f_y(x,y)). \end{aligned}}$$

So, the directional derivative satisfies

$$\boxed{D_{\vec{u}} f(\vec{x}_0) = \vec{\nabla} f(\vec{x}_0) \cdot \vec{u}}$$

In general, for $f(\vec{x}) = f(x_1, \dots, x_n)$ and a unit vector $\vec{u}$,

$$D_{\vec{u}} f(\vec{x}) = \vec{\nabla} f(\vec{x}) \cdot \vec{u}$$

where $\vec{\nabla} f = (f_{x_1}, f_{x_2}, \dots, f_{x_n})$.

Ex. Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at $(2,-1)$ in the direction $\vec{v} = 3\vec{i} + 4\vec{j}$.

Gradient of f?

$$\vec{\nabla} f(x,y) = (\underbrace{2xy^3}_{= f_x(x,y)}, \underbrace{3x^2y^2 - 4}_{= f_y(x,y)})$$

Find $\vec{n}$ unit vector in direction of $\vec{v}$:

$$\vec{n} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{v}}{5} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}.$$

$$\Rightarrow \quad D_{\vec{n}} f = \vec{\nabla} f(x,y) \cdot \vec{n}$$

$$= \frac{6}{5} x y^3 + \left(\frac{12}{5} x^2 y^2 - \frac{16}{5} \right)$$

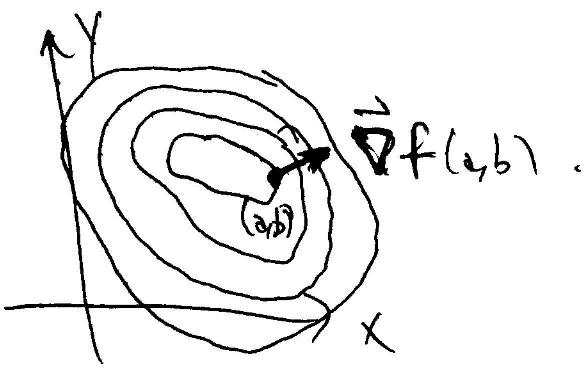
$$\Rightarrow \quad D_{\vec{n}} f \Big|_{(x,y) = (2,-1)} = -\frac{12}{5} + \left(\frac{48}{5} - \frac{16}{5} \right)$$

$$= 4.$$

Significance of gradient vector

Thm. For differentiable f, the maximum value of $D_{\vec{n}} f(\vec{x})$ is $|\vec{\nabla} f(\vec{x})|$ and it occurs when $\vec{n}$ has the direction of $\vec{\nabla} f(\vec{x})$!

Translation: For a surface $f(x,y)$, the gradient gives the direction of steepest ascent (and tells you how steep it is there!).



Pf. of thm. $D_{\vec{n}} f = \vec{\nabla} f \cdot \vec{n}$

$$= |\vec{\nabla} f| \underbrace{|\vec{n}|}_{=1} \cos\theta$$

$$= |\vec{\nabla} f| \cos\theta$$

This is greatest when $\theta = 0$!

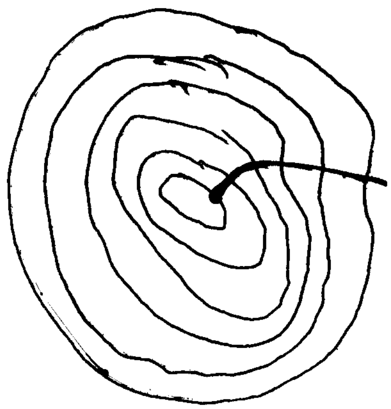
$\Rightarrow$ $D_{\vec{n}} f$ is maximized when $\vec{n} = \dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$

in which case $D_{\vec{n}} f = |\vec{\nabla} f|$!

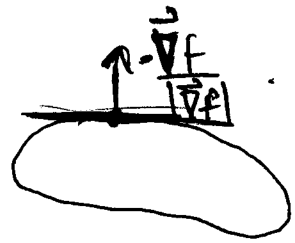
Note that the direction and magnitude of steepest descent is when we take $\vec{u} = -\dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$

in which case $D_{\vec{u}} f = -|\vec{\nabla} f|$.

<u>Remark</u>:



curve of steepest descent/ascent always perpendicular to level curves of function!



level curve.