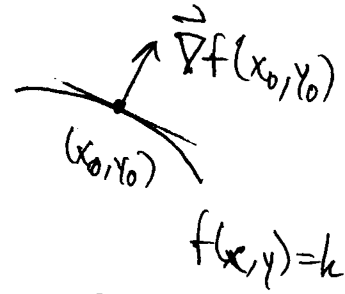
Last time, I claimed that the gradient vector $\vec{\nabla} f$ is always perpendicular to the level surfaces of f. Let's show this:



Suppose $f(x(t), y(t), z(t)) = k$ (constant), i.e., parametric curve

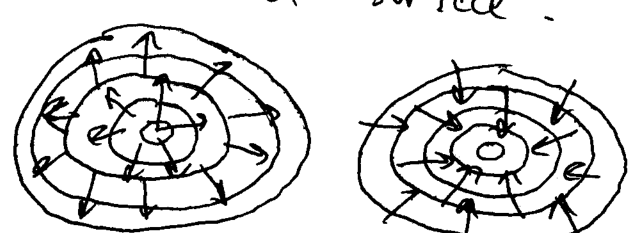
$\vec{r}(t) = (x(t), y(t), z(t))$ lies on surface $f=k$.

Then

$$0 = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \vec{\nabla} f \cdot \vec{r}\,'(t)$$

$\Rightarrow$ Since $\vec{r}\,'(t)$ is always tangent to $r(t)$ and therefore tangent to the level surface $f=k$, so $\vec{\nabla} f$ is perpendicular to level surface!



trough. hill

Ex. Find tangent plane to surface

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3.$$

at point $P(-2,1,3)$.

Let $f(x,y,z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$

Surface corresponds to level surface $f = 3$.

Then surface $f = 3$ has normal vector

$\vec{n} = \vec{\nabla} f(-2,1,-3)$ at the point $P(-2,1,3)$.

$$= \left(\frac{x}{2}, 2y, \frac{2z}{9}\right)\Big|_{(-2,1,+3)} = \left(-1, 2, -\frac{2}{3}\right).$$

So the tangent plane is $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

with $\vec{n} = \left(-1, 2, -\frac{2}{3}\right)$ and $\vec{r}_0 = (-2,1,3)$.

Max. and min. of multivariable fcn's (15.7)

Suppose $f(x,y)$ has continuous second partial derivatives. Then the (a,b) is a

local max $\iff$ $f(a,b) \geq f(x,y)$ for (x,y) in some small disk centered at (a,b)

local min $\iff$ $f(a,b) \leq f(x,y)$ for (x,y) in some small disk centered at (a,b)

global (absolute) max. $\iff$ $f(a,b) \geq f(x,y)$ for all $(x,y) \in D$

global (absolute) min. $\iff$ $f(a,b) \leq f(x,y)$ "

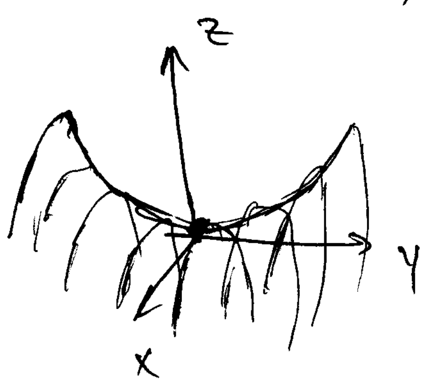
Def. (a,b) is a <u>critical point</u> of f if $f_x(a,b) = 0$ <u>and</u> $f_y(a,b) = 0$ (equivalent to $\vec{\nabla} f = \vec{0}$ at (a,b)), or one of these does not exist.

<u>Basic result</u>: All local max./min. are critical points of f.

<u>Ex.</u> Find all critical points of $f(x,y) = y^2 - x^2$.

$$f_x = -2x = 0 \quad \text{when} \quad x = 0$$
$$f_y = 2y = 0 \quad \text{when} \quad y = 0.$$

$\Rightarrow$ $(0,0)$ only critical point.



But not a local max/min!
Call this a <u>saddle point</u>.

Second derivatives test.

Define the matrix of second partial derivatives of f (called the Hessian)

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

Let $D = D(x,y) = \det H(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

$$= f_{xx}(x,y) f_{yy}(x,y) - (f_{xy}(x,y))^2$$

Then, if (a,b) is a critical point of f,

i) $D > 0$ and $f_{xx}(a,b) > 0 \Rightarrow$ local min.

ii) $D > 0$ and $f_{xx}(a,b) < 0 \Rightarrow$ local max.

iii) $D < 0 \Rightarrow$ saddle point

iv) $D = 0 \Rightarrow$ no information, could be anything.

<u>Ex.</u> $f(x,y) = y^2 - x^2$, $(0,0)$ saddle ?

$f_{xx}(0,0) = -2$

$f_{yy}(0,0) = 2 \quad \Rightarrow \quad D = D(0,0)$

$f_{xy}(0,0) = 0 \qquad\qquad = -2\cdot 2 - 0^2$

$= -4$

$\Rightarrow D < 0 \Rightarrow$ <u>saddle</u>.

<u>Ex.</u> Find and classify all critical points of $f(x,y) = 3x - x^3 - 2y^2 + y^4$

$f_x = 3 - 3x^2 \qquad (= 0$ when $x = -1$ or $x = +1)$

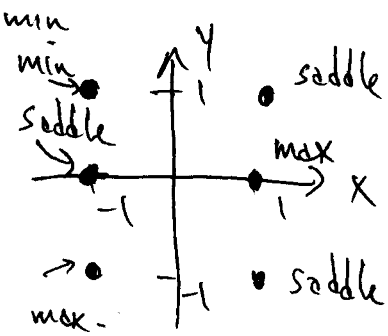
$f_y = -4y + 4y^3 \qquad (= 0$ when $y = 0$ or $y = -1$ or $y = +1)$

six critical points :

$D = -24 < 0$ <u>saddle</u> $\rightarrow (-1, 0)$ $\qquad (-1,-1)$ local max. $\qquad (-1, 1)$ local min

$D = 24 > 0 \rightarrow (1, 0)$ $\qquad (1,-1)$ saddle $\qquad (1,1)$ saddle

$f_{xx} = -6 < 0 \Rightarrow$ <u>local max.</u>



$$\left.\begin{array}{l} f_{xx} = -6x \\ f_{yy} = -4 + 12y^2 \\ f_{xy} = 0 \end{array}\right\} \Rightarrow D(x,y) = 24x - 72xy^2$$

Global max. and min.

Assume f continuous on a closed, bounded set D. Then f achieves its global max./min. in D.

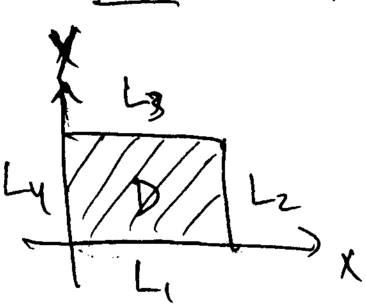
How to find global extrema?

1) Find values of f at critical points in D

2) Find all extrema of f on boundary of D

3) Largest value of f from previous two steps is the global max. (similarly for global min.)

Ex. $f(x,y) = x^2 - 2xy + 2y$ on



$D = \{(x,y) : 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$.

1) $f_x = 2x - 2y$ $\quad$ ($=0$ when $x = y$)

$f_y = -2x + 2$ $\quad$ ($=0$ when $x = 1$)

$\Rightarrow$ critical point at $(1,1) \in D$

$$\boxed{f(1,1) = 1}$$

2) on L_1: i.e, $0 \leq x \leq 3$, $y = 0$

$f = f(x, 0) = x^2$, $x \in [0,3]$

max. on L_1 is $f(3,0) = 9$
min. on L_1 is $f(0,0) = 0$.

on L_2: i.e, $x = 3$, $0 \leq y \leq 2$.

$f = f(3, y) = 9 - 4y$

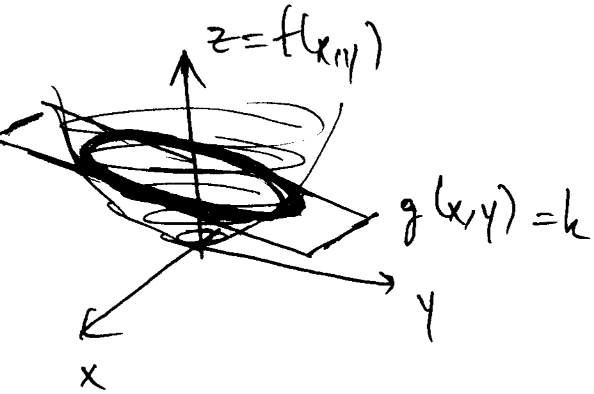
max on $L_2 = f(3,0) = 9$
min on $L_2 = f(3,2) = 1$.

on L_3:
max is 4
min is 0

on L_4 max is 4
min is 0

$\Rightarrow$ global max $= f(3,0) = 9$
global min $= f(0,0) = f(2,2) = 0$.

Lagrange multipliers (15.8)



Suppose we want to find the max. or min. of some function $f(x,y)$ subject to constraint

$$g(x,y) = k.$$

Derivation #1:

Suppose $g(x(t), y(t)) = k$

i.e., parametric curve $\vec{r}(t) = (x(t), y(t))$

lies on curve $g = k$

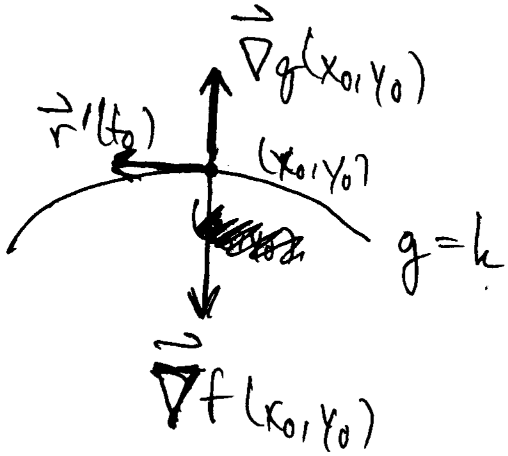
Q: What are extrema of f on $g=k$?

Let $h(t) = f(x(t), y(t))$. To find extrema find t_0 s.t. $h'(t_0) = 0$, i.e.,

$$0 = h'(t_0) = \left.\frac{df}{dt}\right|_{t=t_0} = \left.\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right|_{t=t_0}$$

$$= \vec{\nabla} f \cdot \vec{r}\,'(t) \Big|_{t=t_0}$$

$$= \vec{\nabla} f \underbrace{(x_0, y_0)}_{(x(t_0),\, y(t_0))} \cdot \vec{r}\,'(t_0)$$



Now recall from beginning of lecture that $\vec{\nabla} g$ is everywhere perpendicular to curve $g = k$.

$\Rightarrow$ i.e., $\vec{\nabla} g(x_0, y_0)$ is also perpendicular to curve $g = k$!

Lagrange multiplier.

$$\Rightarrow \boxed{\begin{array}{l} \vec{\nabla} f(x_0, y_0) = \lambda \vec{\nabla} g(x_0, y_0) \\ \text{for some } \lambda \in \mathbb{R} \text{ when} \\ f \text{ is extremized on curve } g = k! \end{array}}$$

<u>Method.</u>

Assuming that $\vec{\nabla} g \neq \vec{0}$ on $g = k$

1) Find all (x, y, z) and λ s.t.

4 eqns. / 4 unknowns.
$$\begin{cases} \vec{\nabla} f(x, y, z) = \lambda \vec{\nabla} g(x, y, z) \\ g(x, y, z) = k. \end{cases}$$

$$f_x(x,y,z) = \lambda g_x(x,y,z)$$
$$f_y(x,y,z) = \lambda g_y(x,y,z)$$
$$f_z(x,y,z) = \lambda g_z(x,y,z)$$

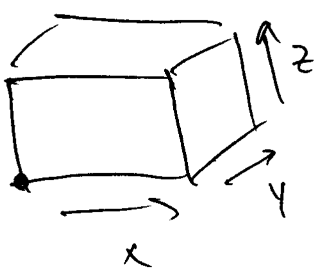
2) Evaluate f at (x, y, z) from part 1 to find max. and min.

Ex. Suppose a box has one vertex at the origin and diagonal vertex lies on surface

$$x^2 + 2y^2 + 3z^2 = 6$$

What is max. volume of box?

maximize $V(x,y,z) = xyz$ subject to



constraint $\underbrace{x^2 + 2y^2 + 3z^2}_{g(x,y,z)} = 6$.

$g(x,y,z) = x^2 + 2y^2 + 3z^2$

1) $\vec{\nabla} V(x,y,z) = (yz, xz, xy)$

$\vec{\nabla} g(x,y,z) = (2x, 4y, 6z)$

$$\vec{\nabla} V = \lambda \vec{\nabla} g \Rightarrow \left.\begin{aligned} yz &= 2\lambda x \\ xz &= 4\lambda y \\ xy &= 6\lambda z \\ x^2 + 2y^2 &+ 3z^2 = 6 \end{aligned}\right\}$$

$\Rightarrow x^2 = 2y^2 = 3z^2 \Rightarrow x = \pm\sqrt{2},\ y = \pm 1,\ z = \pm\sqrt{\frac{2}{3}}$

max. $V = \frac{2}{\sqrt{3}}$.