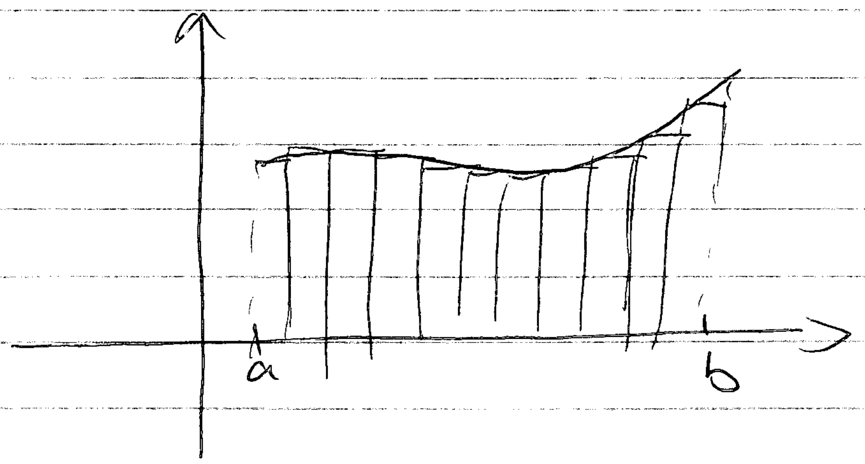
16.1 Double Integrals over Rectangles

The basic idea:

for functions of one variable we had

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(\tilde{x}_j)\,\Delta x$$

where this was the sum of a bunch of small rectangles taking the function value

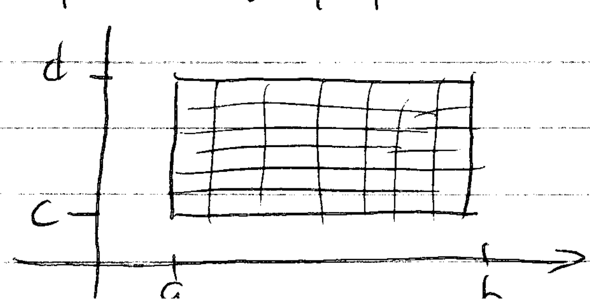


in 2 (and higher) dimensions the same idea holds we are looking at

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \text{The } \underline{\text{volume}}$$

between the surface and the x-y plane

cut up the x-y plane into rectangles of size $\Delta x \Delta y$



then pick some point in each rectangle by some rule
- ex: center, bot-left corner, a random number generator

then the volume of that piece is

$$f(x^*, y^*)\,\Delta x\,\Delta y$$

and the volume is $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y$

the double integral is

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \lim_{m,n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta x\,\Delta y$$

Note regarding notation

take note of the order of $dy\,dx$ or $dx\,dy$. They tell you which variable corresponds to which integral limits. An easy way to remember is to draw parentheses

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

also written as $\iint_R f(x,y)\,dA$

R is the region that you are integrating over

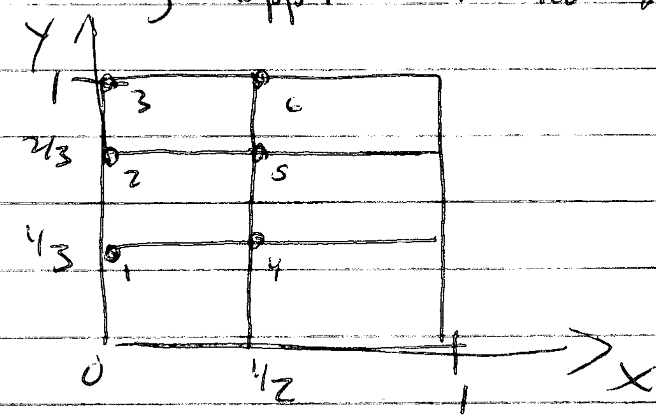
Using Riemann sums

Estimate $\iint_R xy\, dA \qquad R = [0,1]\times[0,1]$

with $m=2 \quad n=3$

using upper left and midpoint rules

upper left

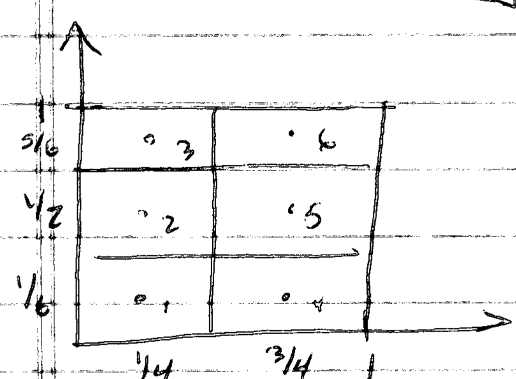


$$\iint_R xy\, dA \approx \underset{1}{0\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)} + \underset{2}{0\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)} + \underset{3}{0\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)}$$

$$+ \underset{4}{\tfrac{1}{6}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)} + \underset{5}{\tfrac{1}{3}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)} + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{3}\right)$$

$$= \boxed{1/6}$$

middle



$$\iint_R xy\, dA \approx \underset{1}{\tfrac{1}{24}\left(\tfrac{1}{6}\right)} + \underset{2}{\tfrac{1}{8}\left(\tfrac{1}{6}\right)}$$

$$+ \underset{3}{\tfrac{5}{24}\left(\tfrac{1}{6}\right)} + \underset{4}{\tfrac{3}{24}\left(\tfrac{1}{6}\right)}$$

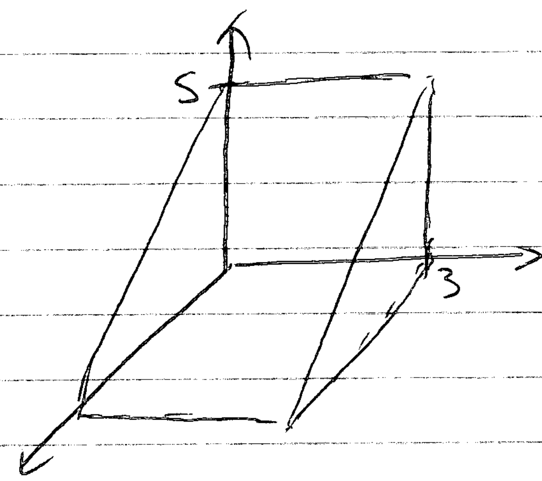
$$+ \tfrac{3}{8}\left(\tfrac{1}{6}\right) + \tfrac{15}{24}\left(\tfrac{1}{6}\right)$$

$$= \tfrac{1}{6}\left(\tfrac{36}{24}\right) = \boxed{\tfrac{1}{4}}$$

example if what you are integrating is some known positive solid, you can use that info

$$\iint_R (5-x)\,dA \qquad R=\{(x,y)\mid 0\le x\le 5,\ 0\le y\le 3\}$$

picture



this can be thought of as a ~~cube~~ rectangular prism, cut in half

sides: 5, 3, 5 $\qquad V_{prism}=75$

$V_{shape}=\boxed{\dfrac{75}{2}}$

note: we can only do this if the surface is above the x-y plane

16.2 Iterated Integrals

Computing via Riemann Sums is a pain, but the fundamental Theorem of Calculus can help us

let's look at this again

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

if we can compute

$$\int_c^d f(x,y)\,dy$$ while keeping x fixed

we can then integrate this answer with respect to x

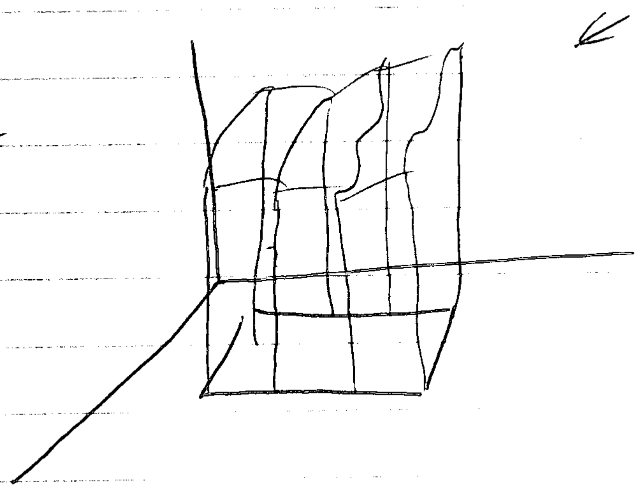
$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

we can also switch the order

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

~~ ~~

picture



we are finding areas of slices

then integrating them

Theorem (Fubini)

if $f(x,y)$ is continuous then

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Advantage: can pick an easier path

example

$$\cancel{\int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx}$$

$$\int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx$$

integrate by parts

$u = y \qquad dv = \sin(xy)$

$du = 1 \qquad v = -\frac{\cos(xy)}{x}$

$$= \int_1^2 \left(-\frac{y}{x}\cos xy \Big|_0^\pi + \int_0^\pi \frac{\cos(xy)}{x}\,dy \right) dx$$

$$= \int_\phi^2 \left(-\frac{\pi}{x}\cos(\pi x) + \frac{\sin(xy)}{x^2}\Big|_0^\pi \right) dx$$

$$= \int_\phi^2 -\frac{\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}\,dx$$

IBP

$u = -\frac{1}{x} \qquad dv = \pi\cos(\pi x)$

$du = 1/x^2 \qquad v = \sin(\pi x)$

$$= -\frac{\sin(\pi x)}{x}\Big|_\phi^2 - \int_0^2 \frac{1}{x^2}\sin\pi x\,dx + \int_0^2 \frac{\sin\pi x}{x^2}\,dx$$

$$= \frac{-\sin(2\pi)}{2} + \frac{\sin \pi}{1} = \boxed{0}$$

other direction

$$\int_0^{\pi}\int_1^{2} y\sin(xy)\,dx\,dy$$

$$= \int_0^{\pi} -\cos(xy)\Big|_1^2 \, dy$$

$$= \int_0^{\pi} \cos(y) - \cos(2y)\, dy$$

$$= \sin(y) - \tfrac{1}{2}\sin(2y)\Big|_0^{\pi}$$

$$= \boxed{0}$$

example

$$\int_1^2 \int_0^3 x^2y - 4xy \, dx \, dy$$

$$= \int_1^2 \frac{x^3y}{3} - 2x^2y \Big|_{x=0}^{x=3} dy$$

$$= \int_1^2 9y - 18y \, dy$$

$$= \int_1^2 -9y \, dy$$

$$= \frac{-9}{2} y^2 \Big|_1^2 = -\frac{9}{2}(4-1) = \boxed{\frac{-27}{2}}$$

note: just as in 1D, we can get "negative" volumes

opposite order

$$\int_0^3 \int_1^2 x^2y - 4xy \, dy \, dx$$

$$= \int_0^3 \frac{x^2y^2}{2} - 2xy^2 \Big|_{y=1}^{y=2} dx$$

$$= \int_0^3 2x^2 - 8x - \frac{x^2}{2} + 2x \, dx$$

$$= \left(\frac{2}{3}x^3 - 3x^2 - \frac{1}{6}x^3\right)\Big|_0^3 = 18 - 27 - \frac{27}{6}$$

$$= -9 - \frac{9}{2}$$

$$= \boxed{\frac{-27}{2}}$$

ex|

$$\iint_R xy\, dA \qquad R = [0,1]\times[0,1]$$

$$= \int_0^1 \int_0^1 xy\, dx\, dy$$

$$= \int_0^1 \frac{x^2 y}{2}\Big|_0^1 dy = \int_0^1 \frac{y}{2}\, dy$$

$$= \frac{y^2}{4}\Big|_0^1 = \boxed{1/4}$$

ex| Find the average value of xy over $R = [-2,3]\times[1/2, 1]$

$$A(R) = 5\left(\frac{1}{2}\right) = 5/2$$

$$\iint_R xy\, dA = \int_{-2}^{3}\int_{1/2}^{1} xy\, dy\, dx$$

$$= \int_{-2}^{3} \frac{xy^2}{2}\Big|_{1/2}^{1} dx$$

$$= \int_{-2}^{3} \frac{x}{2} - \frac{x}{8}\, dx$$

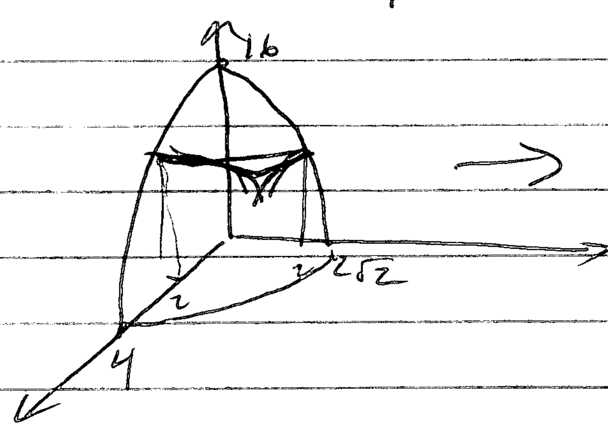
$$= \frac{3}{8}\,\frac{x^2}{2}\Big|_{-2}^{3} = \frac{3}{8}\left(\frac{9}{2} - \frac{4}{2}\right)$$

$$= \frac{15}{16}$$

$$f_{avg} = \frac{1}{A(R)}\iint_R f\, dA = \frac{2}{5}\,\frac{15}{16} = \boxed{\frac{3}{8}}$$

example find the volume of the solid bounded by $x^2+2y^2+z=16$ the planes $x=2,\ y=2$ and the coordinate planes

$z=16-x^2-2y^2$



finding the volume of this

$$V=\int_0^2\int_0^2 16-x^2-2y^2\,dx\,dy$$

$$=\int_0^2 16x-\frac{x^3}{3}-2y^2x\Big|_0^2\,dy$$

$$=\int_0^2 32-\frac{8}{3}-4y^2\,dy$$

$$=32y-\frac{8}{3}y-\frac{4}{3}y^3\Big|_0^2=\boxed{64-\frac{16}{3}-\frac{32}{3}}$$

$$=(48)$$

example

$$\iint_R x^3\cos y + x^3 e^y \, dA \qquad R = [-3,2]\times[-\pi,0]$$

$$= \int_{3}^{2}\int_{-\pi}^{0} x^3(\cos y + e^y)\, dA$$

$$= \int_{-3}^{2} x^3 \int_{-\pi}^{0} \cos y + e^y \, dA$$

you can bring x out just like a constant

$$= \int_{3}^{2} x^3 (\sin y + e^y)\Big|_{-\pi}^{0} \, dx$$

$$= \int_{-3}^{2} x^3 (1 - e^{-\pi})\, dx$$

$$= \boxed{\left(\frac{1-e^{-\pi}}{4}\right)(2^4 - 2^{-3})}$$

Definition Average Value

the average value of f over a region R is

$$f_{av} = \frac{1}{A(R)} \iint_R f(x,y)\, dA$$

What is the approximate average value of the previous problem?

Because $R = [0,1] \times [0,1]$

$A(R) = 1$ so

$f_{av} \approx \frac{1}{6}$ or $\frac{1}{4}$

depending on the rule

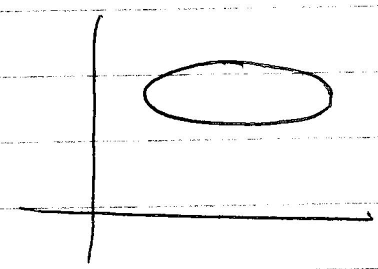
16.3 Double Integrals over general regions

Recall: from last time

Procedure: for double integrals over non-rectangular regions:

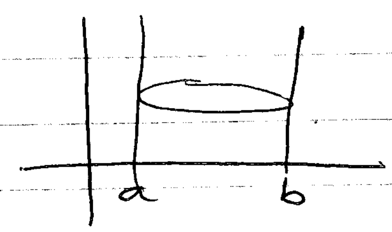
1. Draw the region

ex



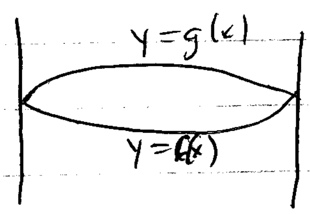
2. Pick one direction (x or y) to draw two ~~straight~~ horizontal/vertical lines that bound the figure

this is the outer integral



$$\int_a^b \int f(x,y)\,dy\,dx$$

3. find the equation of the curve that forms the top + bottom (or left + right) of the figure



this should be something of the form

$y = g(x)$ if y is the inner variable

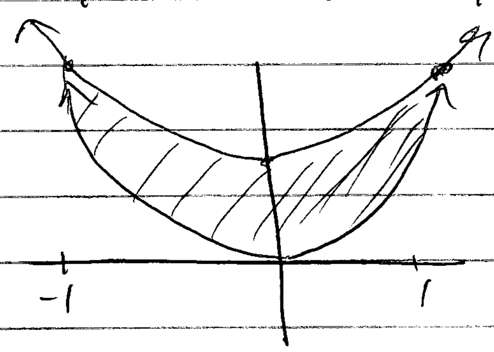
$x = h(y)$ if x is the inner variable

$$\int_a^b \int_{h(x)}^{g(x)} f(x,y)\,dy\,dx$$

example find $\iint_D x+2y\,dA$ where D

is the region bounded by
$y=2x^2$ and $y=1+x^2$

1. Picture



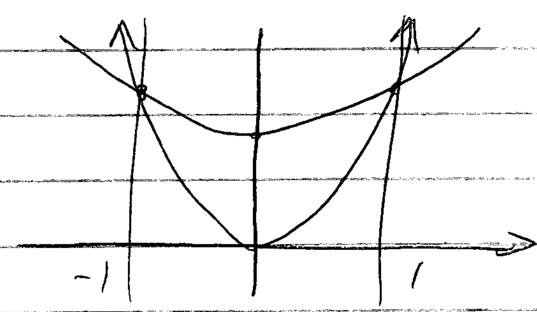
where do they intersect?

$2x^2=1+x^2$

$x^2=1$

$x=\pm 1$

2. Pick bounding lines



$$\int_{-1}^{1}\int x+2y\,dy\,dx$$

3. find eqns of top & bottom

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y\,dy\,dx$$

$$=\int_{-1}^{1} xy+y^2\Big|_{2x^2}^{1+x^2}dx=\int_{-1}^{1} x(1+x^2)+(1+x^2)^2-2x^3-4x^4\,dx$$

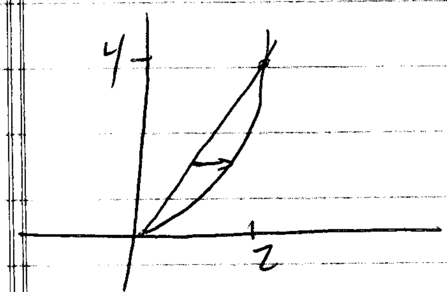
$$=\int_{-1}^{1} x+x^3+1+2x^2+x^4-2x^3-4x^4\,dy$$

$= x + \frac{x^2}{2} - \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{3}{5}x^5 \Big|_{-1}^{1}$

$= 2 + 0 - 0 + \frac{4}{3} - \frac{6}{5}$

ex| find the volume under x^2+y^2 above the region bounded by $y=x^2$ and $y=2x$

do it both ways



Way 1

$$\int_0^2 \int_{x^2}^{2x} x^2+y^2 \, dy \, dx$$

$$= \int_0^2 x^2(2x-x^2) + \frac{1}{3}(8x^3 - x^6) \, dx$$

$$= \int_0^2 2x^3 - x^4 + \frac{8}{3}x^3 - \frac{1}{3}x^6 \, dx$$

$$= \frac{1}{2}x^4 - \frac{1}{5}x^5 + \frac{2}{3}x^4 - \frac{1}{21}x^7 \Big|_0^2$$

$$= 8 - \frac{32}{5} + \frac{32}{3} - \frac{128}{21}$$

Way 2 $$\int_0^4 \int_{y/2}^{\sqrt{y}} x^2+y^2 \, dx \, dy$$

$$= \int_0^4 \frac{1}{3}\left(y^{3/2} - \frac{y^3}{8}\right) + y^2\left(\sqrt{y} - \frac{y}{2}\right) dy$$

$$\cancel{= \int_0^4 \frac{1}{3} y^{3/2} - \frac{y^3}{24} + \frac{y^3}{2}}$$

$$= \int_0^4 \frac{1}{3}y^{3/2} + y^{5/2} - \frac{13}{24}y^3 \, dy$$

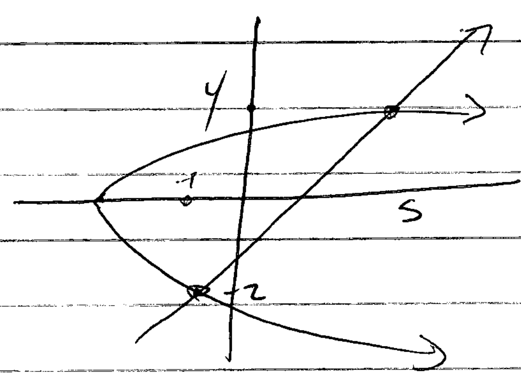
$$= \frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4 \Big|_0^4$$

$$= \frac{2}{15}(32) + \frac{2}{7}(128) - \frac{13}{96}(256)$$

ex ~~what it was bounded by~~

$f(x,y) = xy$

$D =$ the region between $y = x - 1$ and $y^2 = 2x+6$



intersect?

$(x-1)^2 = 2x + 6$

$x^2 - 2x + 1 = 2x + 6$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = \frac{y^2}{2} - 3$

2 ways

easier

$$\int_{-2}^{4} \int_{\frac{y^2}{2}-3}^{y+1} xy \, dx \, dy$$

$$= \int_{-2}^{4} \frac{y}{2}\left((y+1)^2 - \left(\frac{y^2}{2} - 3\right)^2\right) dy$$

$$= \int_{-2}^{4} \frac{y}{2}\left(y^2 + 2y + 1 - \frac{y^4}{4} + 3y^2 - 9\right) dy$$

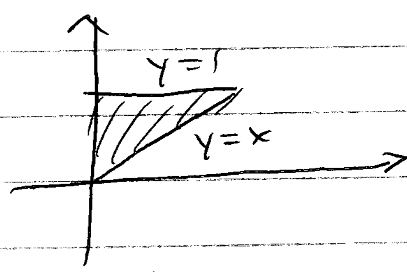
$$= \int_{-2}^{4} -\frac{y^5}{8} + 2y^3 + y^2 - 4y \, dy$$

$$= -\frac{y^6}{48} + \frac{1}{2}y^4 + \frac{1}{3}y^2 - 2y^2 \Big|_{-2}^{4}$$

ex1 $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

impossible to integrate as written

draw the picture



change order of integration

$$\int_0^1 \int_0^y \sin(y^2)\,dx\,dy$$

can int w.r.t. x

$$= \int_0^1 y\sin(y^2)\,dy \qquad \text{subsitute}$$

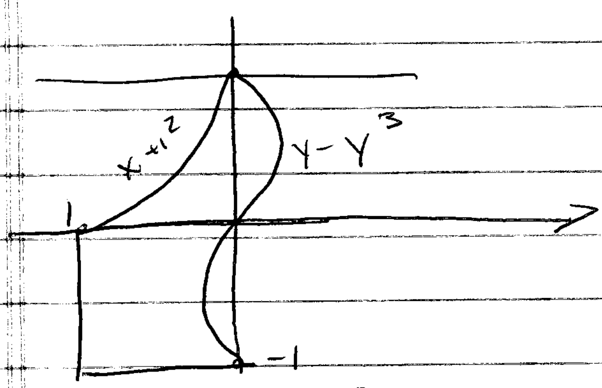
$$= \frac{-\cos(y^2)}{2}\Big|_0^1 = \frac{-\cos(1)}{2} + \frac{1}{2}$$

fact If $D = D_1 \cup D_2$ and D_1, D_2 do not overlap (except possibly on boundaries) then

$$\iint_D f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$$

ex $\iint y\,dA$

between $x = y - y^3$ and $y = (x+1)^2$, $x \geq -1$ $y \geq -1$



go Left to right

$$\int_0^1 \int_{\sqrt{y}-1}^{y-y^3} y\,dx\,dy$$

$$= \int_0^1 y^2 - y^4 - y^{3/2} + y\,dy$$

$$= \boxed{\frac{1}{3} - \frac{1}{5} - \frac{2}{5} + \frac{1}{2}}$$

estimating if $m \leq f(x) \leq M$ on D

then

$$m\,A(D) \leq \iint_D f(x) \leq M\,A(D)$$

ex estimate $\iint_D e^{-(x^2+y^2)}$ over the rectangle $[-1,1]\times[-1,1]$

$f = e^{-(x^2+y^2)}$ max is at $(0,0)$ of $1 = M$

min is a furthest point $(1,1)$

(or any other corner)

$m = e^{-2}$

estimate $\quad 4e^{-2} \le \iint_D e^{-(x^2+y^2)}\, dA \le 4$

note: we could also get a coarser estimate by noting

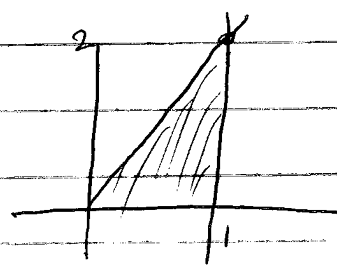
$e^{-(x^2+y^2)} \ge 0$ because exponential is always positive

Then we would have

$$0 \le \iint_D e^{-(x^2+y^2)}\, dA \le 4$$

example $\quad \iint_D \sin^4(x+y)\, dA \quad$ where D = the triangle bounded by

$y=0 \quad x=1 \quad y=2x$



$\leftarrow$ Area $= \frac{1}{2}(1)(2) = 1$

a simple estimate

$-1 \le \sin(\text{anything}) \le 1$

$0 \le \sin^4(\text{anything}) \le 1$

so

$$0 \le \iint_D \sin^4(x+y)\, dA \le 1$$

can we get a better one?

Solve a max-min problem on D

when is $\nabla f = 0$?

$$f_x = f_y = \cancel{4\cos^3(x+y)\sin x} + 4\sin^3(x+y)\cos(x+y)$$

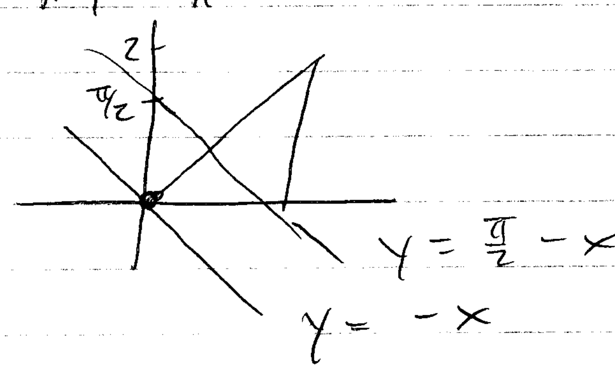
0 when

$x + y = \frac{\pi}{2} + n\pi \quad \Rightarrow y = -x + \frac{\pi}{2} + n\pi$

or $x + y = n\pi \qquad y = -x + n\pi$

↑ critical points are <u>lines</u> in this case

try $n = 0$



on $y = \frac{\pi}{2} - x \qquad f = \sin\left(\frac{\pi}{2}\right) = 1 = \underline{max}$

at $(0,0)$, which is in D,

$f = \sin(0) = 0 = \underline{min}$

so this is the best estimate we can do with this trick