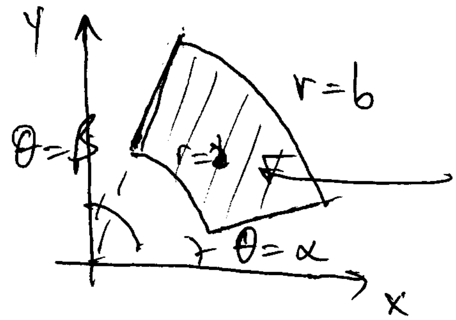
11/30/10

Double integrals in polar coordinates (16.4)

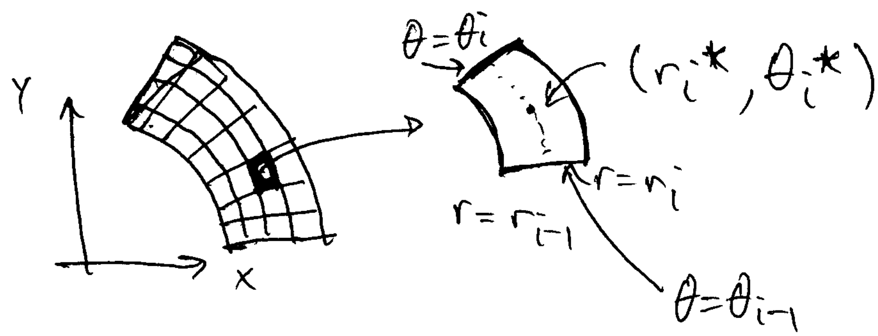
Often, it's easier to evaluate the double integral $\iint_R f(x,y)\,dx\,dy$ using polar coordinates.

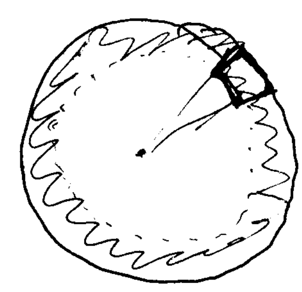


R = "polar rectangle".

Here, we let $x = r\cos\theta$, $y = r\sin\theta$ $\left(\text{so, } r^2 = x^2+y^2,\ \theta = \tan^{-1}\left(\frac{y}{x}\right)\right)$

Computing the Riemann sum in polar coordinates:





$$\Delta A_i = \text{area of } i\text{th element}$$

$$= \frac{\theta_i - \theta_{i-1}}{2\cancel{\pi}} \cdot \cancel{\pi}\left(r_i^2 - r_{i-1}^2\right)$$

$$= (\theta_i - \theta_{i-1})\cdot(r_i - r_{i-1})\cdot \tfrac{1}{2}(r_i + r_{i-1})$$

$$= \Delta\theta_i \cdot \Delta r_i \cdot r_i^*$$

$$\iint_R f(x,y)\,dA = \lim_{\substack{n,m \\ \to\infty}} \sum_{i=1}^{n}\sum_{j=1}^{m} f(r_i^*\cos\theta_j^*,\ r_i^*\sin\theta_j^*)\,\Delta A_i$$

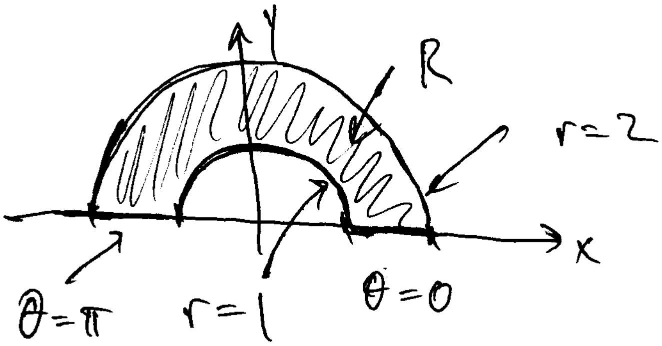
$$= \int_\alpha^\beta \int_a^b f(r\cos\theta,\ r\sin\theta)\ r\ dr\,d\theta$$

don't forget this extra factor of r!

(this factor is called the <u>Jacobian</u> of the transformation $(x,y) \to (r,\theta)$).

<u>Ex.</u> Find $\iint_R (x+y)^2\,dA$ where

R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.



$$\iint_R (x+y)^2\,dA$$

$$= \iint_R \left(\underbrace{x^2+y^2}_{} + 2xy\right)dA$$

$$= \int_0^{\pi}\int_1^{2} \left(r^2 + 2r^2\cos\theta\sin\theta\right) r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^{2} r^3\left(1 + 2\cos\theta\sin\theta\right) dr\,d\theta$$

$$= \int_0^{\pi}\left(1+2\cos\theta\sin\theta\right)\underbrace{\left(\int_1^{2} r^3\,dr\right)}\,d\theta$$

$$= \left[\frac{r^4}{4}\right]_1^2$$

$$= \frac{15}{4}.$$

$$= \frac{15}{4}\int_0^{\pi}\left(1 + 2\cos\theta\sin\theta\right)d\theta$$

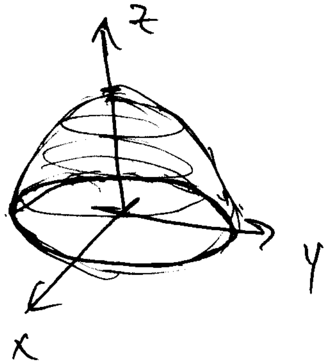
$$= \frac{15}{4}\pi + \underbrace{\frac{2\cdot 15}{4}\int_0^{\pi}\cos\theta\,\underset{\underset{u}{\parallel}}{\sin\theta}\,d\theta}$$

$$\begin{pmatrix} u = \sin\theta \\ du = \cos\theta\,d\theta \end{pmatrix}$$

$$= \cancel{\frac{2\cdot 15}{4}\int_0^{0} u\,du} \; 0$$

$$= \boxed{\frac{15}{4}\pi}.$$

Ex. Find the volume bounded between the paraboloid $\underbrace{z = 1 - x^2 - y^2}$ and x-y plane.



First, when is $z \geq 0$? When $x^2+y^2 \leq 1$.

The paraboloid intersects the x-y plane when $x^2+y^2=1$.

So, $V = \iint_R z \, dA$, with

$R = \{(x,y) : x^2+y^2 \leq 1\}$.

$$= \iint_R (1 - x^2 - y^2)\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= 2\pi \cdot \frac{1}{4} = \boxed{\frac{\pi}{2}}.$$

Remark: Much more difficult to do in rectangular coord's.

<u>Ex.</u> Show that the area under the <u>Gaussian density</u> $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is 1.

$$A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

$$A^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx \right) \cdot \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, dy \right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2}{2}} e^{-\frac{y^2}{2}} \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} \, dx \, dy$$

$$= \int_{0}^{2\pi} \left(\int_{0}^{\infty} \frac{1}{2\pi} e^{-\frac{r^2}{2}} \, r \, dr \right) d\theta$$

$$= \cancel{2\pi} \cdot \frac{1}{\cancel{2\pi}} \int_{0}^{\infty} e^{-\frac{r^2}{2}} \, r \, dr$$

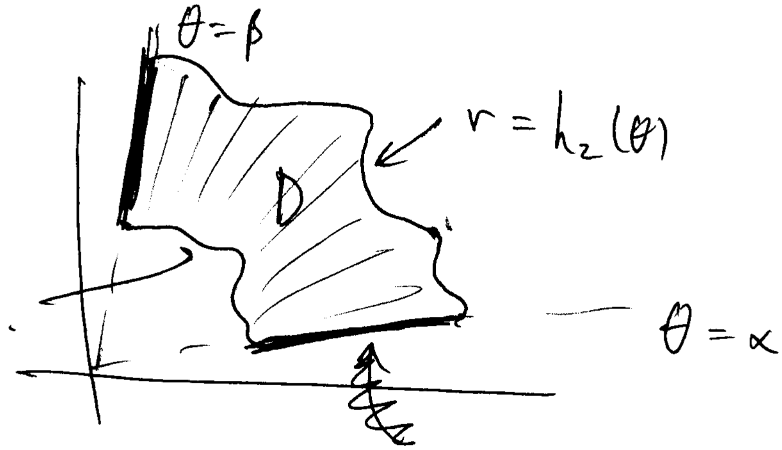
$u = \frac{r^2}{2}$

$du = r \, dr$

$$= \int_{0}^{\infty} e^{-u} \, du = 1.$$

$$\Rightarrow \boxed{A = 1.}$$

In general:

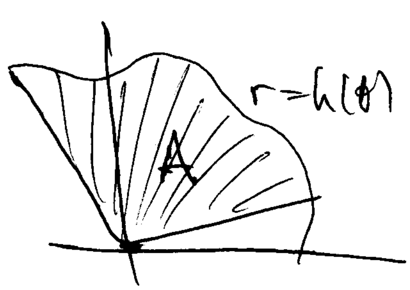


$$D = \{(r,\theta) : \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

For continuous f,

$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

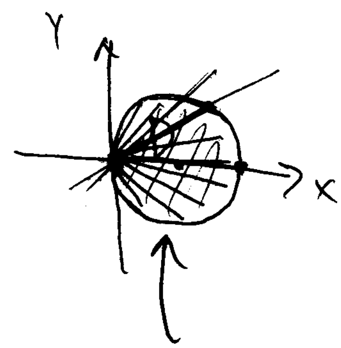
Note: This reduces to the formula for the area bounded by a polar curve $r = h(\theta)$ when $f \equiv 1$, $h_1(\theta) = 0$, $h_2(\theta) = h(\theta)$



$$\Rightarrow A = \int_\alpha^\beta \int_0^{h(\theta)} 1 \cdot r\, dr\, d\theta$$

$$= \int_\alpha^\beta \frac{h^2(\theta)}{2}\, d\theta.$$

Ex. Find $\iint_D f(x,y)\,dA$ where



$D = \{(x-1)^2 + y^2 \leq 1\}$ and

$f(x,y) = x\,|y|$.

$r = 2\cos\theta$

$\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\iint_D f\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\cos\theta\,|r\sin\theta|\,r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r\cos\theta \cdot r\sin\theta \;\; r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \underbrace{\left(\int_0^{2\cos\theta} r^3\,dr\right)}_{= \left[\frac{r^4}{4}\right]_0^{2\cos\theta}} \cos\theta\,\sin\theta\,d\theta$$

$$= 8\int_0^{\frac{\pi}{2}} \left(\cos^4\theta \;\cos\theta\;\sin\theta\right) d\theta$$

$$= -\frac{8}{3}\int_0^{\frac{\pi}{2}} \underbrace{\cos^3\theta}_{= u} \;\underbrace{(-3\cos^2\theta \cdot \sin\theta)\,d\theta}_{du}$$

$$= -\frac{8}{3}\int_1^0 u\, du$$

$$= \boxed{\frac{4}{3}}.$$

Change of variable in multiple integrals, Jacobians (16.9)

For functions of a single variable, the change of variables $x = g(u)$ implies

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,\frac{dx}{du}\cdot du$$

Jacobian.

with $a = g(c)$, $b = g(d)$.

What to do for $\iint_D f(x,y)\,dx\,dy$?

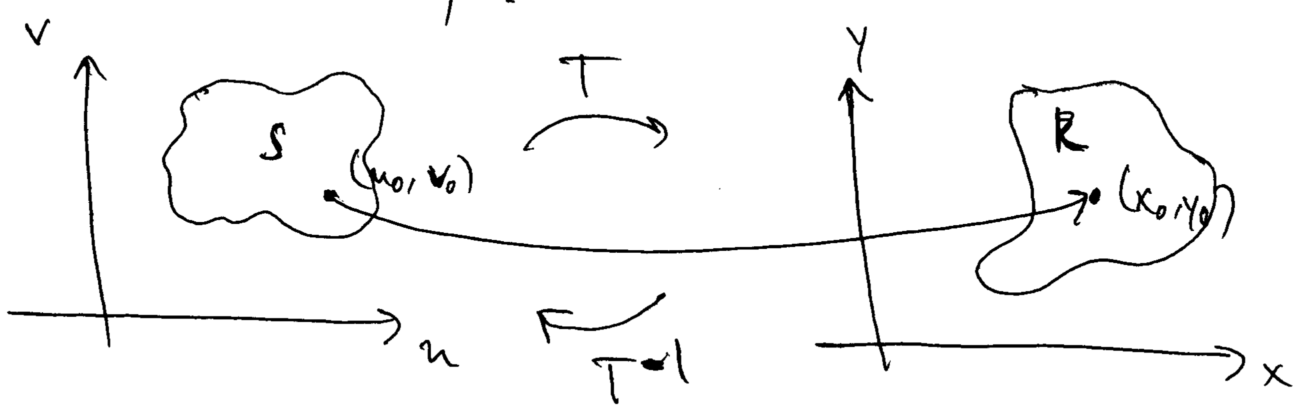
Transformations

Suppose $T : (u,v) \longrightarrow (x,y)$

given by $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$

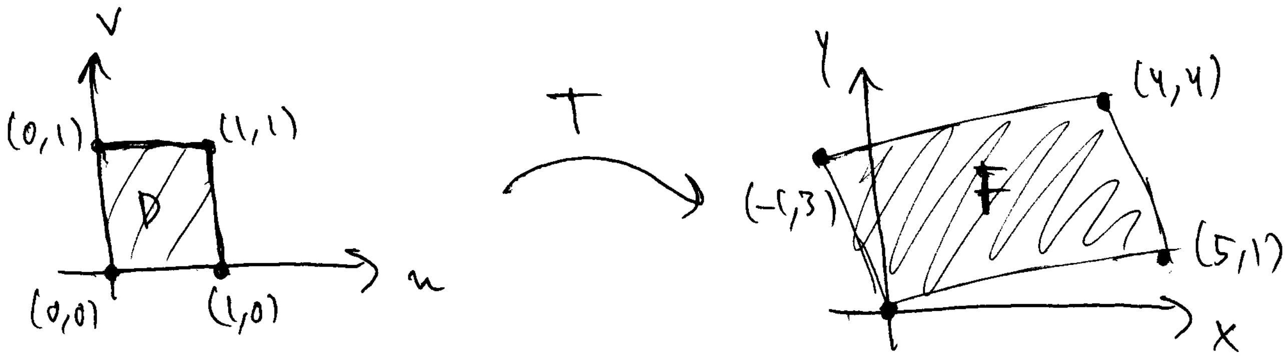
where g, h are continuously differentiable.

Assume T is one-to-one (meaning that no two points have the same image under T).



<u>Ex.</u> Find the image of the unit square $D = \{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$ under the transformation $T(u,v) = (x,y)$ given by

$$\begin{cases} x = 5u - v \\ y = u + 3v. \end{cases}$$



$T(1,1) = (4,4)$ $T(0,1) = (-1,3)$

$T(1,0) = (5,1)$ $T(0,0) = (0,0)$

$\boxed{12/02/10}$

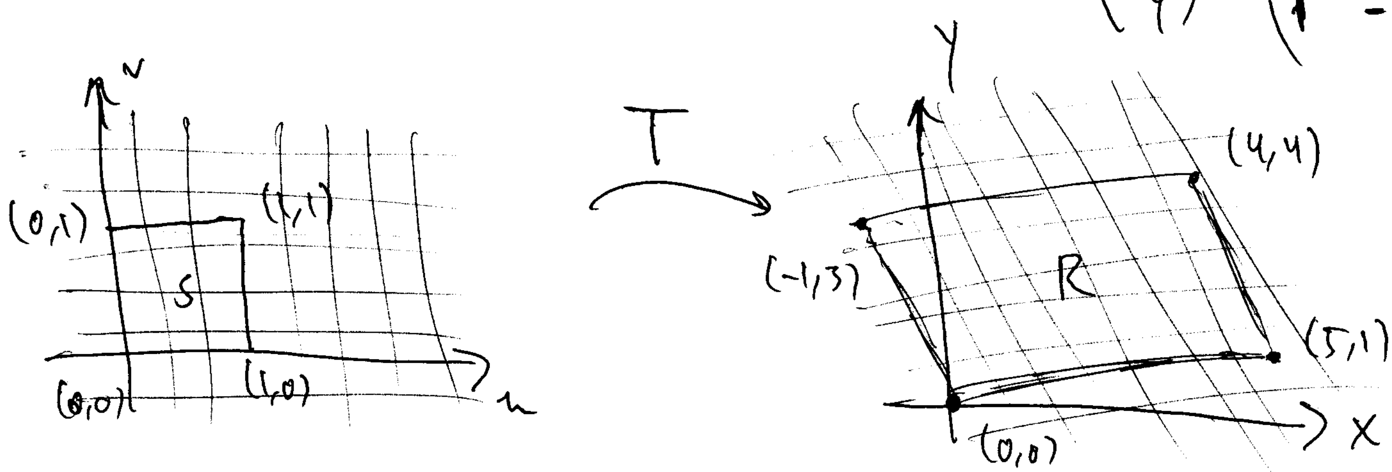
Last time, we introduced <u>transformations</u>

$$T : (u, v) \longrightarrow (x, y), \quad \text{given} \quad \begin{cases} x = g(u,v) \\ y = h(u, v) \end{cases}$$

We say T is a <u>linear transformation</u> if g and h are linear functions of u, v.

<u>Ex.</u> $T : \begin{cases} x = 5u - v \\ y = u + 3v \end{cases}$ linear transformation

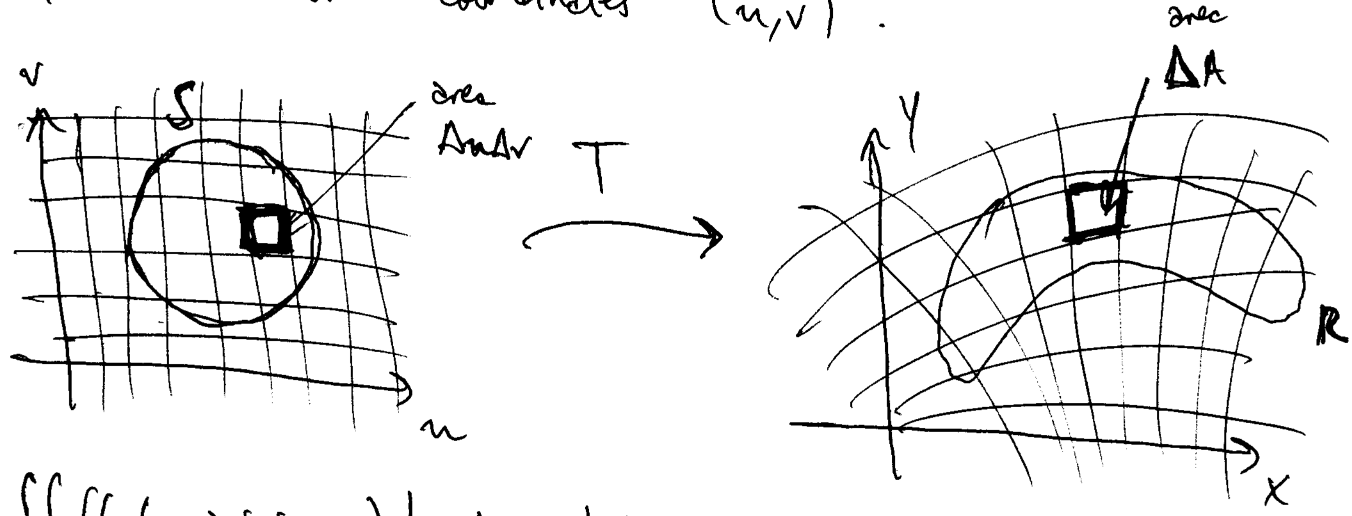
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$



<u>Remark</u>: Linear transformations take parallelograms to parallelograms.

<u>Note</u>: Locally, every differentiable transformation looks linear.

Suppose we want to evaluate $\iint_R f(x,y)\, dA$ in terms of coordinates (u,v).

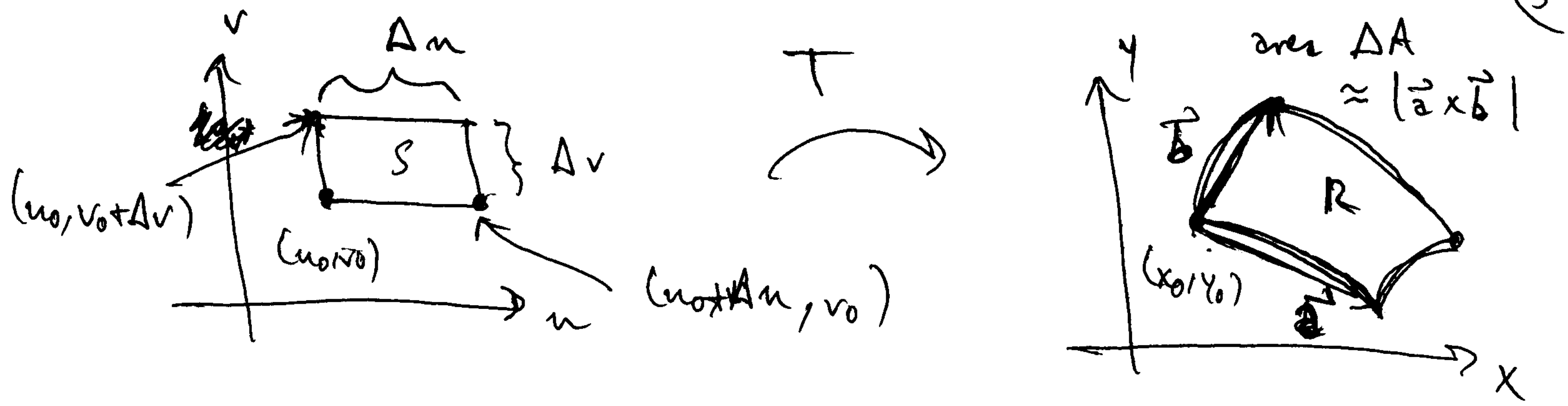


$$\iint_S f(g(u,v),h(u,v))\, \underline{|J(u,v)|}\, du\, dv \qquad \iint_R f(x,y)\, dA$$

$$\qquad\qquad \shortparallel \qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$$

$$\lim \sum_i \sum_j f(g(u_i,v_j), h(u_i,v_j))\, \underline{|J(u_i,v_j)|}\, \Delta u \Delta v \;=\; \lim \sum_i \sum_j f(x_i,y_j)\, \Delta A$$

<u>Q:</u> How does the area of an infinitesimal rectangle with sides Δu, Δv, change under the transformation $T: (u,v) \longrightarrow (x,y)$? What is $J(u,v)$?



The position of $T(u,v)$ is

$$\vec{r}(u,v) = \underset{\substack{\parallel \\ x}}{g(u,v)}\,\vec{i} + \underset{\substack{\parallel \\ y}}{h(u,v)}\,\vec{j}.$$

The image R of S has sides approximated by displacement vectors

$$\vec{a} = \vec{r}(u_0+\Delta u, v_0) - \vec{r}(u_0, v_0) \approx \Delta u \frac{\partial \vec{r}}{\partial u}$$

$$\vec{b} = \vec{r}(u_0, v_0+\Delta v) - \vec{r}(u_0, v_0) \approx \Delta v \frac{\partial \vec{r}}{\partial v}$$

$$\Delta A \approx |\vec{a}\times\vec{b}| = \Delta u\,\Delta v \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right|$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} \right) \times \left(\frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} \right)$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{bmatrix}$$

$$= \underbrace{\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}}_{J(u,v)} \vec{k}$$

<u>Def.</u> The <u>Jacobian</u> of the transformation $T : (u,v) \to (x,y)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

$$\Rightarrow \ dx\,dy = dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

(analogue of $dx = \frac{dx}{du}\,du$ for integrals in one-dim.)

<u>Remark:</u> The absolute value of the Jacobian tells us how areas scale under the transformation

$(u,v) \to (x,y)$.

To recap:

Thm. If $T : (u,v) \to (x,y)$ is a differentiable, one-to-one transformation that maps S to R, and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$, then

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

Ex. Find the Jacobian of the linear transformation $\begin{cases} x = 5u - v \\ y = u + 3v \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

$$= 16.$$

Note: Jacobian of linear transformations are everywhere constant.

Ex. Find Jacobian of the transformation

$$\begin{cases} x = uv \\ y = u/v \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \det\begin{bmatrix} v & \frac{1}{v} \\ u & -\frac{u}{v^2} \end{bmatrix}$$

$$= -\frac{uv}{v^2} - \frac{u}{v} = -\frac{2u}{v}$$

Ex. Find Jacobian of $\begin{cases} x = e^{s+t} \\ y = e^{s-t} \end{cases}$

$$\frac{\partial(x,y)}{\partial(s,t)} = \det\begin{bmatrix} e^{s+t} & e^{s-t} \\ e^{s+t} & -e^{s-t} \end{bmatrix}$$

$$= -e^{s+t}e^{s-t} - e^{s-t}e^{s+t}$$

$$= \boxed{-2e^{2s}}.$$

Ex. Rederive formula for double integral in polar coordinates.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

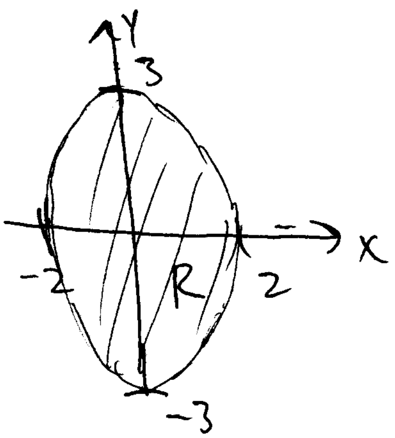
$$= \det\begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = r > 0$$

$$\Rightarrow \quad \iint_R f(x,y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, \underbrace{r}_{= \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|}\, dr\, d\theta.$$

Ex. Evaluate $\iint_R x^2 \, dA$ where



$$R = \{(x,y) : 9x^2 + 4y^2 \leq 36\}$$

Step 1: Find a transformation $(u,v) \to (x,y)$ that maps a simple region S to the given region R.

$$9x^2 + 4y^2 \leq 36 \iff \frac{1}{4}x^2 + \frac{1}{9}y^2 \leq 1$$

$$\iff \Big(\underbrace{\frac{x}{2}}_{=u}\Big)^2 + \Big(\underbrace{\frac{y}{3}}_{=v}\Big)^2 \leq 1$$

$\Rightarrow$ make transformation $\begin{cases} x = 2u \\ y = 3v \end{cases} \Rightarrow \dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = 6$

This tells us that S is unit circle in (u,v) plane.

Step 2: Evaluate integral.

$$\iint_R x^2\, dA = \iint_S (2u)^2 \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{=6} du\, dv$$

$$= \iint_S 24 u^2\, du\, dv$$

where $S = \{(u,v) : u^2+v^2 \le 1\}$.

$u = r\cos\theta$
$v = r\sin\theta$

$$= 24 \int_0^{2\pi}\int_0^1 r^2\cos^2\theta\, r\, dr\, d\theta$$

$$= 24 \int_0^{2\pi} \cos^2\theta \underbrace{\left(\int_0^1 r^3\, dr\right)}_{=\frac{1}{4}} d\theta$$

$$= 6 \int_0^{2\pi} \cos^2\theta\, d\theta$$

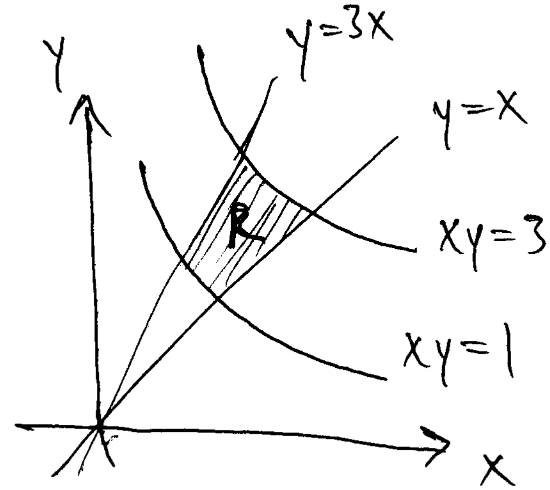
$$= 6 \int_0^{2\pi} \frac{1}{2}\left(1+\cos(2\theta)\right) d\theta$$

$$= 3\left(2\pi + \int_0^{2\pi} \cos(2\theta)\, d\theta\right)$$

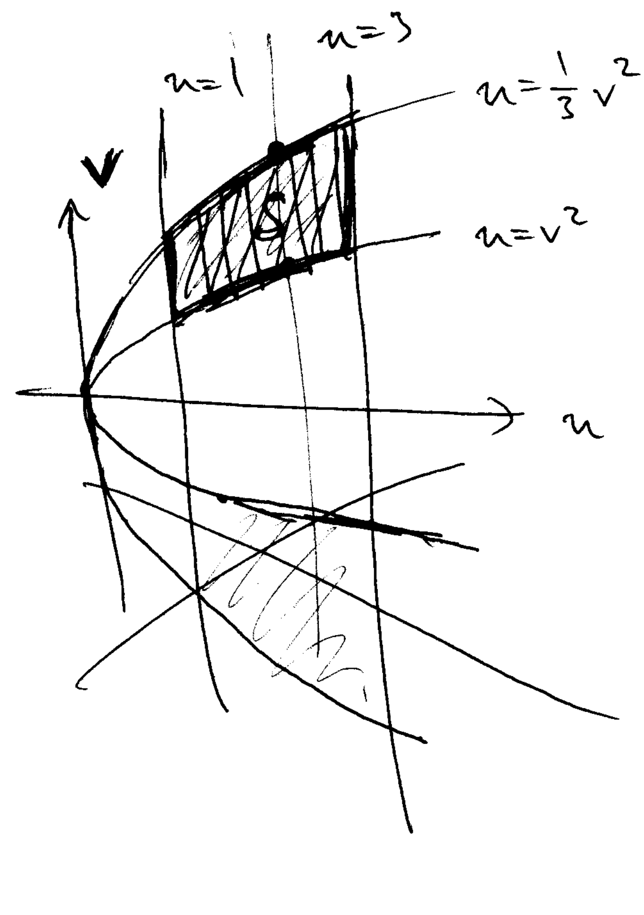
$$= 3\left(2\pi + \cancel{\tfrac{1}{2}\sin(2\theta)\Big]_0^{2\pi}}^{\;0}\right)$$

$$= \boxed{6\pi}.$$

Ex. $\displaystyle\iint_R xy\, dA$ with



Step 1: Use the transformation $(u,v) \to (x,y)$ given by
$$\begin{cases} x = \dfrac{u}{v} \\ y = v \end{cases}$$



$y = x \iff v = \dfrac{u}{v}$

$\iff \boxed{u = v^2}$

$y = 3x \iff v = 3\dfrac{u}{v}$

$\iff \boxed{u = \tfrac{1}{3}v^2}$

$xy = 1 \iff \dfrac{u}{v}\cdot v = 1$

$\iff \boxed{u = 1}$

$xy = 3 \iff \boxed{u = 3}.$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$= \det\begin{bmatrix} \frac{1}{v} & 0 \\ -\frac{u}{v^2} & 1 \end{bmatrix}$$

$$= \frac{1}{v} \neq 0 \quad \text{on } S$$

$$\iint_R xy\, dA = \iint_S u \underbrace{\left|\frac{1}{v}\right|}_{= \left|\frac{\partial(x,y)}{\partial(u,v)}\right|} du\, dv$$

$$= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\, dv\, du$$

$$= \int_1^3 u \left(\int_{\sqrt{u}}^{\sqrt{3u}} \frac{1}{v}\, dv \right) du$$

$$= \int_1^3 u \Big[\ln v \Big]_{\sqrt{u}}^{\sqrt{3u}}\, du$$

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$$= \int_1^3 x \left(\ln\sqrt{3x} - \ln\sqrt{x} \right) dx$$

$$\left(\begin{aligned} &\ln\sqrt{3x} - \ln\sqrt{x} \\ &= \ln\sqrt{3} \\ &= \tfrac{1}{2}\ln 3 \end{aligned} \right)$$

$$= \int_1^3 \left(\tfrac{1}{2}\ln 3 \right) x \, dx$$

$$= \tfrac{1}{2}\ln 3 \left(\frac{3^2}{2} - \frac{1^2}{2} \right)$$

$$= \boxed{2\ln 3}.$$