We have been considering infinite series $\sum_{n=1}^{\infty} a_n$. Recall that $\sum_{n=1}^{\infty} a_n \doteq \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} a_i}_{s_n}$

if r.h.s. exists as a finite limit (convergent series). Otherwise, divergent series.

Now we consider some important examples where the sum exists and can be explicitly written:

Geometric series:

Let $c, r$ be constants, $c \neq 0$.

Consider $c + cr + cr^2 + \dots + cr^{n-1} + \dots$

$$= \sum_{n=1}^{\infty} \underbrace{cr^{n-1}}_{a_n}$$

$r$ is the common ratio.

For which $r$ is this series convergent? Sum?

$\boxed{r=1:}$ $c + c + c + \dots + c + \dots$

$$s_n = \sum_{i=1}^{n} a_i = nc \longrightarrow \pm\infty \Rightarrow \text{divergent!}$$

$\boxed{r \neq 1:}$

$$s_n = c + \boxed{cr + cr^2 + \dots + cr^{n-1}}$$

$$r s_n = \boxed{cr + cr^2 + cr^3 + \dots +} \, cr^n$$

$$s_n - rs_n = c - cr^n$$

$$\Rightarrow s_n = \frac{c - cr^n}{1-r} = \frac{c(1-r^n)}{1-r}$$

Recall that $\lim_{n\to\infty} r^n = \begin{cases} +\infty & \text{if } r>1 \\ 0 & -1<r<1 \\ \text{does not exist (divergent)} & r \leq -1 \end{cases}$

$$\Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{c(1-r^n)}{1-r}$$

$$= \frac{c(1-\lim_{n\to\infty} r^n)}{1-r}$$

$$= \frac{c}{1-r} \quad \text{if } -1<r<1$$

and is <u>divergent</u> otherwise.

$$\boxed{\begin{array}{l} \underline{\text{Geometric series}}: \\ \sum_{n=1}^{\infty} cr^{n-1} \text{ is convergent with sum } \frac{c}{1-r} \\ \text{if } |r|<1 \text{ and is divergent if } |r| \geq 1. \end{array}}$$

Ex. $\sum_{n=0}^{\infty} (-1)^n 2^{2n} 5^{1-n}$ convergent? Sum?

$$\sum_{n=0}^{\infty} (-1)^n 2^{2n} 5^{1-n} = \overbrace{(-1)^0 2^{2\cdot 0} 5^{1-0}}^{=5} + \underbrace{\sum_{n=1}^{\infty} (-1)^n 2^{2n} 5^{1-n}}$$

$$(-1)^n 2^{2n} 5^{1-n} = (-1)^n (2^2)^n \left(\frac{1}{5}\right)^n \cdot 5$$

$$= \underbrace{\left((-1)(2^2)\left(\frac{1}{5}\right)\cdot 5\right)}_{=c} \cdot \left(\underbrace{-1 \cdot 2^2 \cdot \frac{1}{5}}_{=r}\right)^{n-1}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n 2^{2n} 5^{1-n}$$

$$= \sum_{n=1}^{\infty} c r^{n-1} \quad \text{with}$$

$c = -4$

$r = -\frac{4}{5}$.

$\Rightarrow$ convergent series with sum $\frac{c}{1-r} = \frac{-4}{1+\frac{4}{5}}$

$= \frac{-20}{9}$

Final answer is $\boxed{5 - \frac{20}{9}.}$.

**Ex.** Write $4.7\overline{51} = 4.7\underline{51}\,51\,51\,51\ldots$ as a fraction.

$$4.7\overline{51} = \frac{47}{10} + \underbrace{\frac{51}{10^3} + \frac{51}{10^5} + \frac{51}{10^7} + \cdots}$$

$$= \frac{47}{10} + \underbrace{\sum_{n=1}^{\infty} 51 \cdot \frac{1}{10^{2n+1}}}$$

Finish at home!

$$\sum_{n=1}^{\infty} cr^{n-1}$$

$$51 \cdot \frac{1}{10^{2n+1}} = \frac{51}{10} \cdot \frac{1}{10^{2n}} = \frac{51}{10}\left(\frac{1}{10^2}\right)^n$$

$$= \underbrace{\frac{51}{10} \cdot \frac{1}{10^2}}_{=c} \cdot \left(\underbrace{\frac{1}{10^2}}_{r}\right)^{n-1}$$

$$(a^n = a \cdot a^{n-1})$$

**Ex.** (Telescoping series)

Is $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{1}{n(n+1)}}_{a_n}$ convergent? Sum?

$$s_n = \sum_{i=1}^{n} a_i$$

$$= \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

(Now we use that $\displaystyle\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$.)

$$= \sum_{i=1}^{n} \left( \frac{1}{i} - \frac{1}{i+1} \right)$$

$$= \left( \frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left( \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left( \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \right)$$

$$+ \cdots + \left( \cancel{\frac{1}{n}} - \frac{1}{n+1} \right)$$

$$\boxed{s_n = 1 - \frac{1}{n+1}}$$

$$\lim_{n\to\infty} s_n = 1 - \lim_{n\to\infty} \frac{1}{n+1} = 1.$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1}$$

<u>Ques:</u> How about $\sum_{n=1}^{\infty} (-1)^n = -1 + \cancel{1} - \cancel{1} + \cancel{1} - 1 + 1 \ldots$

Can't assume that this is telescoping!

$$s_n = \sum_{i=1}^{n} (-1)^i = \begin{cases} -1 & \text{if } n \text{ odd} \\ +1 & \text{if } n \text{ even.} \end{cases}$$

$\lim_{n\to\infty} s_n$ does not exist! $\Rightarrow$ <u>divergent</u>.

Now we go back to tests for divergence or convergence.

**Thm.** (tails of summands must vanish)
If $\sum_{n=1}^{\infty} a_n$ is convergent, then
$\lim_{n\to\infty} a_n = 0$.

The converse ($\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ convergent) is <u>not</u> true. Why? For ex., harmonic series $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n}}_{a_n}$ (here, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ but the infinite series does not converge!).

The theorem is equivalent to

**Test for divergence** (first thing to check!):
If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=1}^{\infty} a_n$ is divergent.

**Warning:** If $\lim_{n\to\infty} a_n = 0$, divergence test **does not** say that series converges!

**Ex.** $\sum_{n=1}^{\infty} \frac{e^{-n}}{e^{-n}+e^{-2n}}$ convergent?

$$a_n = \frac{e^{-n}}{e^{-n}+e^{-2n}} = \frac{1}{1+e^{-n}} \longrightarrow 1 \text{ as } n\to\infty.$$

$\Rightarrow$ Sum is divergent since $a_n \not\to 0$ as $n\to\infty$!

**Thm.** (Rearrangement of series)

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent and $c$ is a constant, then we have that the following series are convergent:

(i) $\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$.

<u>Ex.</u> $\sum_{n=1}^{\infty} \left( \frac{1}{e^n} + \frac{1}{n(n+1)} \right)$ convergent? Sum?

$\sum_{n=1}^{\infty} \frac{1}{e^n}$ is a geometric series with $c = \frac{1}{e}$

$r = \frac{1}{e} < 1$

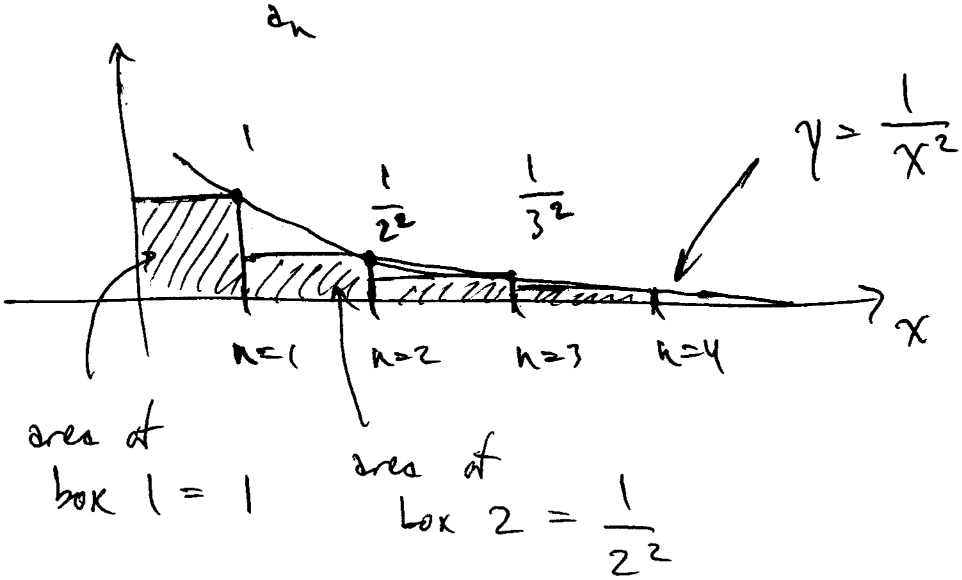
$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ ~~is~~ a telescoping series

with sum 1.

## Integral test and estimating sums (12.3)

Consider $\sum_{n=1}^{\infty} a_n$ with $a_n \geq 0$.

<u>Ex.</u> $\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{a_n}$ convergent? Yes.



Notice that curve $y = \frac{1}{x^2}$ lies above graph of boxes. $\Rightarrow$ area under $y = \frac{1}{x^2}$ greater than sum of areas of boxes,

i.e., $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2}\,dx$.

So, if $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent

then $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ is convergent, with

sum less than $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$.

We know that $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ converges

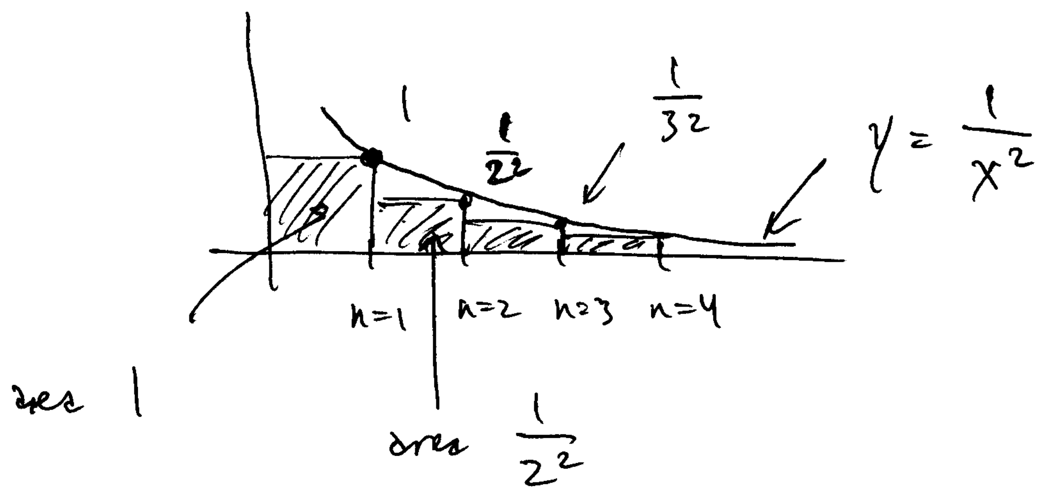
with value 1

$\Rightarrow \displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges with sum $\leq 1$.

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$.

(Exact value is actually $\frac{\pi^2}{6}$, $\leq 2$.)

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent ?



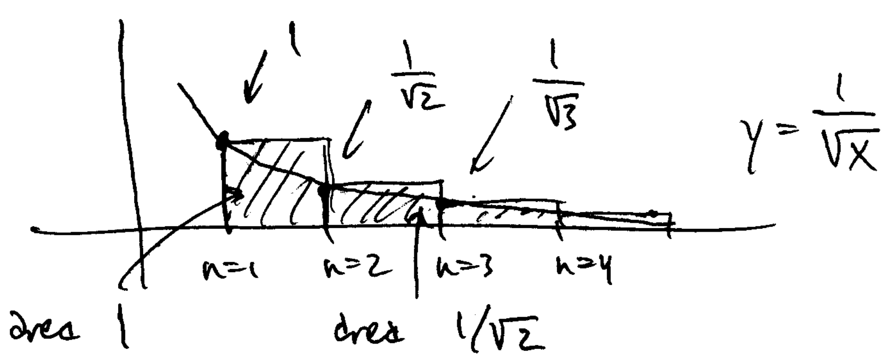

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2} < \underbrace{\int_1^{\infty} \frac{1}{x^2}\,dx = 1}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \underbrace{\sum_{n=2}^{\infty} \frac{1}{n^2}} < 2$$

convergent.

$\left( \sum_{n=10}^{\infty} a_n \text{ convergent} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ convergent} \right)$

Ex. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ convergent ?



$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \underbrace{\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx}$$

this is <u>divergent to $\infty$</u> !

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ divergent to } \infty.$$

<u>Integral test</u>:

Let $f$ be <u>continuous</u>, <u>positive</u>, and <u>decreasing</u> function on $[1, \infty)$, and let $a_n = f(n)$.

Then,

(i) $\int_1^{\infty} f(x)\,dx$ convergent $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$ convergent.

(ii) $\int_1^{\infty} f(x)\,dx$ divergent $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n$ divergent

<u>Warning</u>: This only tells us whether the series is convergent or divergent, it doesn't give us its value!

**Ex.** (p- series)

For what values of $p$ is $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

Integral test implies

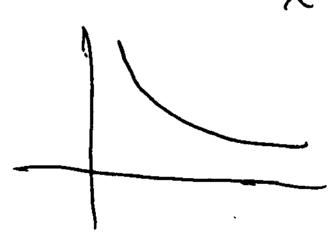
$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ convergent} \iff \underbrace{\int_1^{\infty} \frac{1}{x^p}\,dx \text{ convergent}}$$

only converges for $p > 1$, diverges for $p \leq 1$.

$$\boxed{\begin{array}{l}\underline{\text{p- series:}} \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ Convergent if } p > 1 \text{ and divergent} \\ \qquad \text{if } p \leq 1.\end{array}}$$

**Ex.** $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)}$ converges?

$$\int_1^{\infty} \frac{1}{x \ln x}\,dx = \underbrace{\int_0^{\infty} \frac{1}{u}\,du}_{\text{diverges!}} \qquad \left(\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}\right)$$



$\Rightarrow$ no, series diverges.

## Estimating sum of series using integrals:

Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. What is

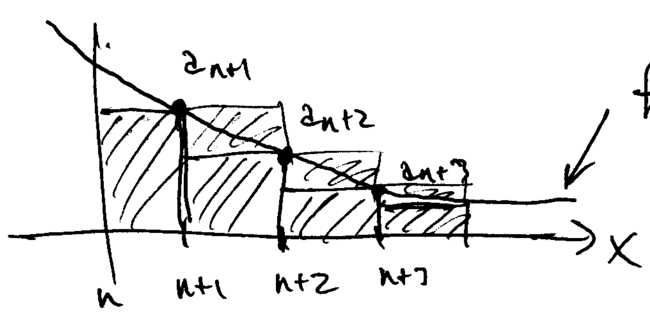
$$s = \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \underbrace{\sum_{i=1}^{n} a_i}_{s_n}$$

$s_n$ (nth partial sum)

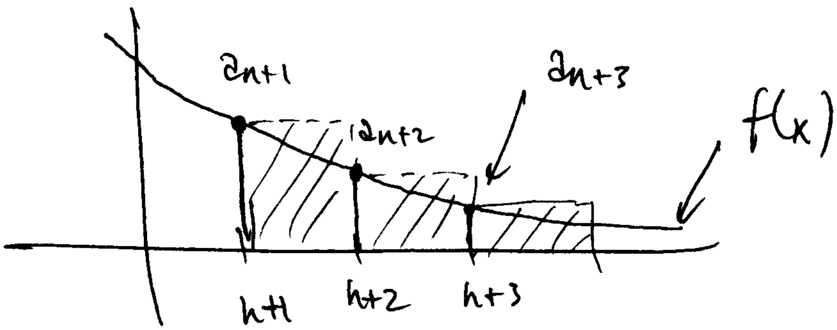
Consider $R_n = s - s_n = \sum_{i=n+1}^{\infty} a_i$

(remainder)

Suppose there is an function $f$ which is continuous, positive, decreasing s.t. $f(n) = a_n$.




$$\Rightarrow \underbrace{\sum_{i=n+1}^{\infty} a_i}_{= R_n} \leq \int_{n}^{\infty} f(x)\, dx$$




$$\Rightarrow \underbrace{\sum_{i=n+1}^{\infty} a_i}_{= R_n} \geq \int_{n+1}^{\infty} f(x)\,dx$$

**Remainder estimate:**

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

## Comparison tests (12.4)

Let's consider the analogue of the comparison test for improper integrals (8.8) for series.

Comparison test:

Suppose $\sum a_n$, $\sum b_n$ are series with positive terms $\{a_n\}$, $\{b_n\}$. Then

(i) $\boxed{a_n \leq b_n \text{ for all } n \text{ and } \sum b_n \text{ convergent}} \Rightarrow \sum a_n$ convergent.

(ii) $\boxed{a_n \geq b_n \text{ for all } n \text{ and } \sum b_n \text{ divergent}} \Rightarrow \sum a_n$ divergent

What should we compare to? Typically, one of these:

- p-series $\left( \sum_{n=1}^{\infty} \frac{1}{n^p} \right.$ converges if $p > 1$, diverges if $\left. p \leq 1 \right)$

- geometric series

  $\left( \sum_{n=1}^{\infty} c r^{n-1} \right.$ converges if $|r| < 1$, diverges if $\left. |r| \geq 1 \right)$

Ex. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{\ln n}{n}}_{a_n}$ convergent?

$a_n = \dfrac{\ln n}{n}$

$b_n = \dfrac{1}{n}$

$\ln n \geq 1$ for $n \geq 3$

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} = \frac{\ln 1}{1} + \frac{\ln 2}{2} + \underbrace{\frac{\ln 3}{3} + \cdots}_{\sum_{n=3}^{\infty} \frac{\ln n}{n}}$$

Compare to $\displaystyle\sum_{n=3}^{\infty} \overbrace{\frac{1}{n}}^{b_n}$.

$$\sum_{n=3}^{\infty} \frac{\ln n}{n} \geq \underbrace{\sum_{n=3}^{\infty} \frac{1}{n}}_{\text{diverges to } \infty!}$$

$\Rightarrow$ series is divergent!

Ex. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ convergent?

$\hookrightarrow$ looks like $\dfrac{1}{2^n}$

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

not true!

$\cancel{\dfrac{1}{2^n-1} \leq \dfrac{1}{2^n}}$ for all $n$, in fact $\dfrac{1}{2^n-1} > \dfrac{1}{2^n}$

For ex.,

$$\frac{1}{2^n-1} \leq \frac{1}{(1.5)^n} \quad \text{for} \quad n \geq 2.$$

Know that $\underbrace{\sum_{n=1}^{\infty} \frac{1}{(1.5)^n}}_{\text{geometric series with } r=\frac{1}{1.5}<1.}$ is convergent

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2^n-1} \text{ is convergent}$$

by comparison test.

<u>Ex.</u> $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n-n^2+7n+1}$ convergent ?

behaves $\frac{1}{2^n}$

<u>Limit comparison test</u> :

Suppose $\sum a_n$ and $\sum b_n$ are series w/ positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where $c$ is finite and $c>0$, then

either **both** series converge or **both** diverge!

**Ex.**

$$a_n = \frac{1}{2^n - n^2 + 7n + 1}$$

$$b_n = \frac{1}{2^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - n^2 + 7n + 1}$$

$$= \lim_{n\to\infty} \frac{1}{1 - \frac{n^2}{2^n} + \frac{7n}{2^n} + \frac{1}{2^n}}$$

$$= 1.$$

So, by limit comparison,

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges} \iff \sum_{n=1}^{\infty} \frac{1}{2^n - n^2 + 7n + 1}$$

# Alternating series (12.5)

Suppose $\{b_n\}$ are positive terms and let $a_n = (-1)^{n-1} b_n$.

Convergence of

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

$$= b_1 - b_2 + b_3 - b_4 + \cdots$$

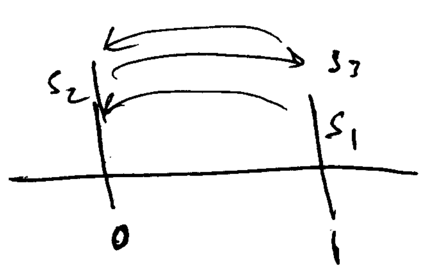
(with $b_n > 0$.).

<u>Ex.</u> $b_n = 1$.

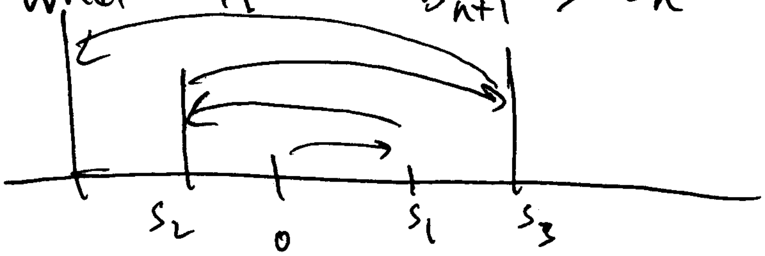
$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot 1 = 1 - 1 + 1 - 1 + \cdots$$

$$s_n = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$\lim_{n\to\infty} s_n$ does not exist, so series <u>diverges</u>!

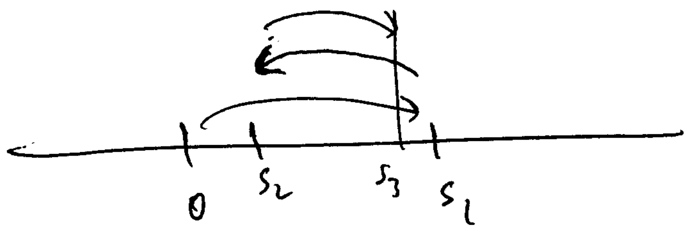


What if $b_{n+1} \geq b_n$ ?




Divergent !

What if $b_{n+1} < b_n$ ?




<u>Alternating series test :</u>

If the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$, $b_n > 0$ satisfies

(i) $b_{n+1} \leq b_n$ for all $n$

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is <u>convergent</u>.

<u>Estimating remainder :</u>

In addition, $|R_n| = |s - s_n| \leq b_{n+1}$

Ex. (Alternating harmonic series)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}.$$

Check: (i) $b_{n+1} \leq b_n$ ?

Yes, since $\frac{1}{n+1} \leq \frac{1}{n}$ for all $n$.

(ii) $\lim_{n\to\infty} b_n = 0$

Yes, since $\lim_{n\to\infty} \frac{1}{n} = 0$.

$\Rightarrow$ convergent series !

And $|R_n| \leq \frac{1}{n+1}$.

# Addendum: Using alternating series test

If $a_n = (-1)^{n-1} b_n$ with $b_n \geq 0$ for all $n$, the alternating series test tells us that

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges if

(i) $b_{n+1} \leq b_n$ (for all $n$ larger than some fixed $N$)

(ii) $\lim_{n\to\infty} b_n = 0$.

Condition (ii) is usually easy to verify.
To check condition (i), we either verify $b_{n+1} \leq b_n$ directly or do the following.
Use the interpolating function $f(x)$ s.t. $f(n) = b_n$ for all $n$. If we can show that $f'(x) \leq 0$ for all $x$ greater than some $M > 0$, this implies (ii).

Ex.

$$\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^5}{2n} \quad \text{convergent?}$$

Alternating series test: We verify conditions (i) and (ii) as follows.

(i): Let $f(x) = \dfrac{(\ln x)^5}{2x}$ so that $f(n) = b_n$. Now note that

$$f'(x) = \frac{1}{2x}\left(5(\ln x)^4 \cdot \frac{1}{x}\right) - \frac{(\ln x)^5}{2x^2}$$

$$= \underbrace{\frac{(\ln x)^4}{2x^2}}_{\text{positive for all } x \geq 1} \underbrace{(5 - \ln x)}_{\text{negative for all } x \geq e^5.}.$$

Therefore, $f'(x) \leq 0$ for all $x \geq e^5$ so $b_{n+1} \leq b_n$ for all $n$ large enough.

(ii) Using L'Hopital's rule repeatedly, we find that $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \dfrac{(\ln x)^5}{2x} = 0.$

So, alternating series test

$$\Longrightarrow \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\ln x)^5}{2x} \text{ converges.}$$

# Absolute convergence, and ratio and root tests

(12.6)

Until now, we have discussed series

$\sum a_n$ where $\{a_n\}$ had some special structure.

In general, what to do?

**Def.** $\sum a_n$ absolutely convergent if the series of absolute values $\sum |a_n|$ is convergent.

**Def.** $\sum a_n$ conditionally convergent if it is convergent, but not absolutely convergent.

**Ex.** $\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} \frac{1}{n^2}}_{a_n}$ $\qquad |a_n| = \frac{1}{n^2}$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ convergent? Yes.

$\Rightarrow$ absolutely convergent.

**Ex.** $\sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} \frac{1}{n}}_{a_n}$ $\qquad |a_n| = \frac{1}{n}$

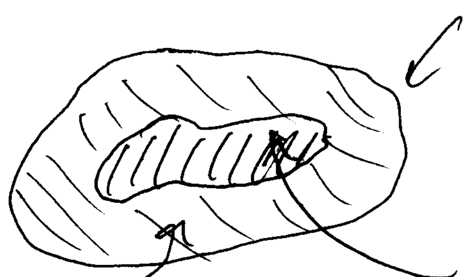
$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent.

But $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ still converges!

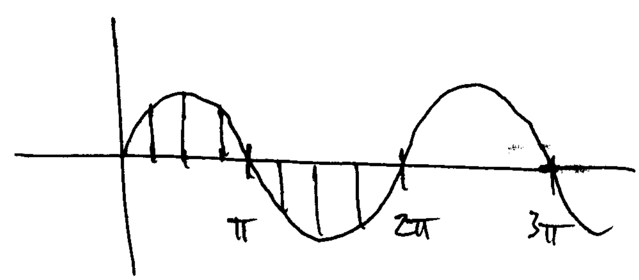
alternating harmonic series

$\Rightarrow$ conditionally convergent.

**Thm.** If $\sum a_n$ absolutely converges, then it converges.




Ex. $\sum_{n=1}^{\infty} \underbrace{\frac{\sin(n)}{n^{3/2}}}_{a_n}$




Is $\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^{3/2}} \right|$ convergent?

$$|a_n| = \left| \frac{\sin(n)}{n^{3/2}} \right| \leq \frac{1}{n^{3/2}}$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty \implies \sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^{3/2}} \right|$ converges.

p-series w/ $p = 3/2$.

$\Rightarrow$ absolutely convergent

$\Rightarrow$ convergent.

Can we test for absolute convergence?

<u>Ratio test:</u>

(i) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ absolutely convergent.}$$

(ii) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = +\infty$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ divergent.}$$

(iii) $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \Rightarrow$ nothing.

(i.e, inconclusive)

<u>Idea of proof of (i):</u>

Comparison to a convergent geometric series...

Suppose $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L = \frac{1}{4} < 1$

Then for some $N$ large enough,

$$\left|\frac{a_{n+1}}{a_n}\right| < \frac{1}{4} + \frac{1}{4}$$ a little bit ... for all $n > N$.

$$= \frac{1}{2}$$

$n = N$

$$\Rightarrow \left|\frac{a_{N+1}}{a_N}\right| < \frac{1}{2} \Rightarrow |a_{N+1}| < \frac{1}{2}|a_N|$$

$n = N+1$

$$\left|\frac{a_{N+2}}{a_{N+1}}\right| < \frac{1}{2} \Rightarrow |a_{N+2}| < \frac{1}{2}|a_{N+1}|$$

$$< \frac{1}{4}|a_N|$$

$\vdots$

$$\Rightarrow |a_{N+k}| < \frac{1}{2}|a_{N+k-1}|$$

$\vdots$

$$< \left(\frac{1}{2}\right)^k |a_N|$$

$r = \frac{1}{2}$

$$\Rightarrow \sum_{n=N+1}^{\infty} |a_n| < \underbrace{\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k |a_N|}_{\text{Converges.}}$$

$c = |a_N|$

$\Rightarrow \sum a_n$ absolutely convergent.

$\blacksquare$

Idea of why (iii) is inconclusive:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \qquad a_n = \frac{1}{n^2}$$

convergent series

Ratio test: $\displaystyle \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+2n+1}$$

$$= 1.$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad a_n = \frac{1}{n}$$

divergent series

Ratio test: $\displaystyle \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$$= \lim_{n\to\infty} \frac{n}{n+1}$$

$$= 1.$$

Ex. $\displaystyle \sum_{n=1}^{\infty} \frac{n!}{n^n}$ convergent?

$n! = n(n-1)(n-2)\cdots 2\cdot 1.$

Ratio test: $\displaystyle a_n = \frac{n!}{n^n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \underbrace{\frac{(n+1)!}{n!}}_{n+1} \cdot \underbrace{\frac{n^n}{(n+1)^{n+1}}}_{\frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1}}$$

$$= \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}$$

$$= \left(\frac{n+1-1}{n+1}\right)^n$$

$$= \left(1 - \frac{1}{n+1}\right)^n$$

$$\left(1-\frac{1}{n+1}\right)^{n+1-1} = \underline{\left(1-\frac{1}{n+1}\right)^{n+1}} \left(1-\frac{1}{n+1}\right)^{-1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n$$

Know: $\lim_{x\to\infty} \left(1 + \frac{c}{x}\right)^x = e^c$.

$$= e^{-1} < 1$$

$\Rightarrow$ <u>absolutely convergent</u>.

We could have also used the comparison test...

$$a_n = \frac{n!}{n^n} = \underbrace{\frac{\cancel{n}(n-1)(n-2)\cdots 2\cdot 1}{\cancel{n}\cdot n \cdot n \cdot n \cdots n\cdot n}}_{n \text{ times}}$$

$$< \frac{2\cdot 1}{n\cdot n} = \frac{2}{n^2}$$

Know that $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges

$\Rightarrow$ convergent by comparison test.

<u>Remark</u>: Ratio test useful when dealing with factorial terms!

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<u>Root test</u>:

(i) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = |a_n|^{1/n} = L < 1$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ absolutely convergent.

(ii) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = +\infty$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ divergent.

(iii) $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1 \Rightarrow$ nothing (i.e., inconclusive)

<u>Remark</u>: Ratio test inconclusive $\Leftrightarrow$ Root test inconclusive

<u>Ex.</u> $\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^{n^2}}_{a_n}$ convergent?

<u>Root test</u>: $$\sqrt[n]{|a_n|} = \left(\left(1+\frac{1}{n}\right)^{n^2}\right)^{1/n}$$

$$= \left(1+\frac{1}{n}\right)^{\frac{n^2}{n}}$$

$$= \left(1+\frac{1}{n}\right)^{n}$$

$$\longrightarrow e^1 \text{ as } n\to\infty.$$

$\Rightarrow$ series <u>divergent</u>.

<u>Remark</u>: <u>Absolutely convergent</u> series behave nicely ~~&~~ compared to <u>conditionally convergent</u> series. (which are delicate).

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Rearrangement of $\sum a_n$ might be:

$$a_3 + a_7 + a_1 + a_4 + a_2 + a_5 + \cdots$$

Convergent?

$\longrightarrow$ If $\sum_{n=1}^{\infty} a_n$ absolutely convergent, with sum $s$, then any rearrangement of $\sum_{n=1}^{\infty} a_n$ is convergent w/ value $s$.

# Strategy for testing series (12.7)

exact
- Geometric series ← $\sum cr^{n-1}$ convergent if $|r|<1$, divergent if $|r|\geq 1$
- p-series ← $\sum \frac{1}{n^p}$ convergent if $p>1$, divergent if $p \leq 1$

first to check.
- Divergence test
  ($\lim_{n\to\infty} a_n \neq 0 \Rightarrow$ divergent)

comparison tests with $\{a_n\}$ positive
- Integral test ← when corresponding integral is easy to evaluate.
- Comparison test
- Limit comparison test ← when comparing against geometric or p-series is possible.

- Alternating series test ← if of the form $\sum (-1)^{n-1} b_n$ ($b_n$ positive.)

tests for arbitrary sign.
- Test for absolute convergence ← if signs arbitrary
- Ratio test ← good for factorials of $n$
- Root test ← good for powers of $n$.

Examples: (p. 758 of book)

1. $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$ $\longrightarrow$ limit comparison test with $\frac{1}{3^n}$

since $\sum \frac{1}{3^n}$ convergent geometric series.

$$\lim_{n\to\infty} \frac{\frac{1}{n+3^n}}{\frac{1}{3^n}} = \lim_{n\to\infty} \frac{3^n}{n+3^n}$$

$$= 1.$$

$\Rightarrow$ convergence.

2. $\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$ $\longrightarrow$ root test.

$$\sqrt[n]{\frac{(2n+1)^n}{n^{2n}}} = \sqrt[n]{\left(\frac{2n+1}{n^2}\right)^n}$$

$$= \frac{2n+1}{n^2} \longrightarrow 0 \quad \text{as } n\to\infty.$$

$\Rightarrow$ absolutely convergent.

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3. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

- divergence test.

$$\lim_{n\to\infty} (-1)^n \frac{n}{n+2}$$

$$\lim_{n\to\infty} \frac{1}{1+\frac{n}{3^n}}$$

$$= \frac{1}{1 + \lim_{n\to\infty} \frac{n}{3^n}}$$

$$\lim_{x\to\infty} \frac{x}{3^x}$$

$$= \lim_{x\to\infty} \frac{1}{3^x \ln 3} = 0.$$

# Power series (12.8)

We define a <u>power series</u> in the real variable $x$ as

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

start at 0

Coefficients of series

We can think of these as polynomials with an infinite number of terms.

<u>Motivation</u>: We will learn how to represent arbitrary functions $f(x)$ on an interval $(a,b)$ ($a, b$ can be infinite) by a power series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n .$$

This gives a natural approximation of $f(x)$ by a polynomial:

$$f(x) \approx \sum_{n=0}^{N} c_n x^n$$

by keeping the first $N$ terms of the series.

→ For what $x$, and what $N$ is this approximation good? How to quantify this?

Before we do this, we must first understand convergence properties of power series.

<u>Def.</u> A <u>power series centered at $a$</u> is the function

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2$$

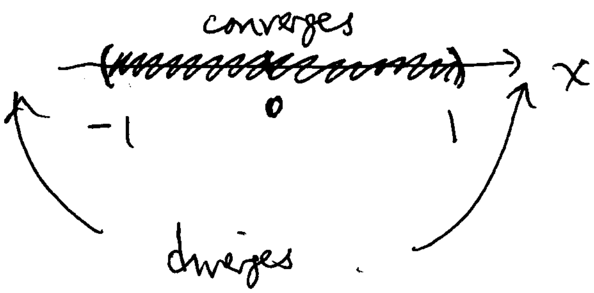
<u>Basic question</u>: For what $x$ is this a convergent series?

<u>Ex.</u> $c_n = 1$, $a = 0$

$$\underbrace{\sum_{n=0}^{\infty} x^n}_{\text{geometric series with } r=x.} = 1 + x + x^2 + x^3 + \cdots$$

$\Longrightarrow$ converges when $|x|<1$ to $\frac{1}{1-x}$

and diverges when $|x| \geq 1$.




$$\left( \begin{array}{l} R = 1 \\ I = (-1, 1). \end{array} \right)$$

<u>Ex.</u> $c_n = n!$ , $a = 0$

$\sum_{n=0}^{\infty} n!\, x^n$ converges for what $x$?

Here, $a_n = n!\, x^n$.

<u>Ratio test</u>: $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$(x \neq 0)$

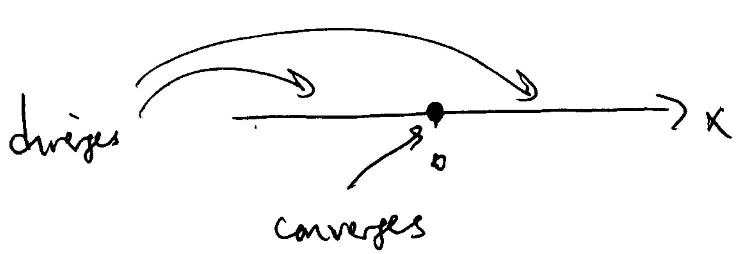
$$= \lim_{n \to \infty} \frac{(n+1)!\, |x|^{n+1}}{n!\, |x|^n}$$

$$= \lim_{n \to \infty} (n+1)\,|x|$$

$$= \infty$$

$\Rightarrow$ <u>diverges</u> for all $x \neq 0$!

converges for $x = 0$.




$$\left( \begin{array}{l} R = 0 \\ I = \{0\} \end{array} \right)$$

Ex. (Bessel function of order 0)

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}, \quad \text{for what } x \text{ is this convergent?}$$

Ratio test: $\quad a_n = \dfrac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} ((n+1)!)^2} \cdot \frac{2^{2n} (n!)^2}{(-1)^n x^{2n}} \right|$$

$$= \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{2^{2n}}{2^{2(n+1)}} \frac{(n!)^2}{((n+1)!)^2} \cdot \frac{x^{2(n+1)}}{x^{2n}} \right|$$

$$= \left| \frac{1}{2^2} \cdot \frac{1}{(n+1)^2} \cdot x^2 \right|$$

$$= \frac{1}{4(n+1)^2} |x|^2 \longrightarrow 0$$

as $n \to \infty$

for all $x$ in $(-\infty, \infty)$

$\Rightarrow$ converges for all $x$ ~~the~~ in $(-\infty, \infty)$

$\xrightarrow{\text{Converges}} x \qquad \left( \begin{array}{l} R = +\infty \\ I = (-\infty, \infty) \end{array} \right).$

<u>Thm.</u> For any power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

(i) series converges only when $x = a$.

(ii) series converges for all $x$ in $(-\infty, \infty)$

(iii) there is some $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

(for $|x-a| = R$, i.e, for $x-a = R$ or $x-a = -R$, have to check by hand on case-by-case basis.)

In (iii), $R$ is called the <u>radius of convergence</u>.

For (i), $R = 0$

(ii), $R = +\infty$

We define the interval of convergence to be all values $x$ for which the series converges.

So, the possible intervals of convergence with $0 \le R \le +\infty$ are:

$(a-R,\ a+R)$, $(a-R,\ a+R]$,

$[a-R,\ a+R)$, $[a-R,\ a+R]$.

<u>Remark</u>: To find radius of convergence $R$, in general use ratio or root test.

For $x = a-R$, $x = a+R$, must use other tests.

<u>Ex.</u> $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ $\qquad a_n = \dfrac{(x-3)^n}{n}$

<u>Root test</u>: $\sqrt[n]{|a_n|} = \left(\dfrac{|x-3|^n}{n}\right)^{1/n}$

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} \qquad = \frac{|x-3|}{n^{1/n}}$$

<u>Ratio test</u>: $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(x-3)^{n+1}}{n+1} \cdot \dfrac{n}{(x-3)^n}\right|$

$$= |x-3| \cdot \frac{n}{n+1} \longrightarrow |x-3|$$

as $n \to \infty$

$\Rightarrow$ converges if $|x-3| < 1$

diverges if $|x-3| > 1$

$\Rightarrow$ converges if $2 < x < 4$

diverges if $x < 2$ or $x > 4$.

What if $x = 2$?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$ converges (alternating harmonic series)

$a_n = (-1)^n b_n$, $b_n = \frac{1}{n}$

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

(ii) $b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_n$. ✓

What if $x = 4$?

$$\sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$ diverges (harmonic series)

$\Rightarrow$ converges if $2 \leq x < 4$
diverges otherwise.

$$\boxed{\begin{aligned} R &= 1 \\ I &= [2, 4) \end{aligned}}$$

Examples (p. 763)

Ex. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{5^n n^5}$ $\qquad a_n = \dfrac{x^n}{5^n n^5}$

Find radius of convergence $R$
interval of convergence $I$.

Ratio test: $\displaystyle \left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{x^{n+1}}{5^{n+1}(n+1)^5} \cdot \frac{5^n n^5}{x^n} \right|$

$$= \left| \frac{x}{5} \cdot \left(\frac{n}{n+1}\right)^5 \right|$$

$$= \frac{1}{5}|x| \left(1 - \frac{1}{n+1}\right)^5$$

$$\longrightarrow \frac{1}{5}|x|$$

$\Rightarrow$ converges if $\frac{1}{5}|x| < 1$

diverges if $\frac{1}{5}|x| > 1$

What if $x = 5$?

$$\sum_{n=0}^{\infty} \frac{5^n}{5^n n^5} = \sum_{n=0}^{\infty} \frac{1}{n^5}$$ converges by p-series test. w/ $p = 5$.

What if $x = -5$?

$$\sum_{n=0}^{\infty} \frac{(-5)^n}{5^n n^5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^5}$$ absolutely converges

$\Rightarrow$ converges.

<u>Ex.</u>

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^n}$$ for what $x$ is this convergent?

$$a_n = \frac{(x-2)^n}{n^n}$$

<u>Root test:</u> $$\sqrt[n]{|a_n|} = \left|\frac{(x-2)^n}{n^n}\right|^{1/n}$$

$$= \frac{|x-2|}{n} \longrightarrow 0$$

as $n \to \infty$.

$\Rightarrow$ converges for all $x$ in $(-\infty, \infty)$.

## Representation of functions as power series (12.9)

$\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$

to $f(x) = \dfrac{1}{1-x}$

$$\boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad -1 < x < 1}$$

<u>Ex.</u> Represent $f(x) = \dfrac{1}{1+x^2}$ in a power series. For which $x$ is this series convergent?

$x = -u^2$

$$\Rightarrow \frac{1}{1+u^2} = \sum_{n=0}^{\infty} (-u^2)^n \quad \text{for} \quad -1 < -u^2 < 1$$

$$\Rightarrow \boxed{\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{for} \quad -1 < x < 1.}$$