M408D (54690/54695/54700), Midterm #1 Solutions

Note: Solutions to the multiple-choice questions for each section are listed below. Due to randomization between sections, explanations *to a version* of each of the multiple-choice problems are given at the end of the document.

	54690	54695	54700
MC#1 (10 pts.)	7. none of them	8. all of them	1. B only
MC#2 (10 pts.)	6. A and B	8. A and B	1. A and B
MC#3 (5 pts.)	2. limit $= 0$	4. limit $= 0$	1. limit $= 0$
MC#4 (5 pts.)	6. $a_n = \frac{n-2}{2^n}$	1. $a_n = \frac{4n-5}{5^n}$	4. $a_n = \frac{n-2}{2^n}$

Question #1 (25 points)

Define the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n\sqrt{\ln n} + \sqrt{n}\ln n}.$$

a) Is the series absolutely convergent? Justify your answer and your use of any test.

Solution: This problem is similar to Question #1 in Quiz 2. We must find whether the series

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n} + \sqrt{n}\ln n}$$

converges or not. To do so, note that the first term in the denominator grows faster than the second term and is therefore the dominant contribution as $n \to \infty$. We can compare this series with

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$$

using the limit comparison test. Let $b_n = \frac{1}{n\sqrt{\ln n} + \sqrt{n}\ln n}$ and $c_n = \frac{1}{n\sqrt{\ln n}}$. We have that

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{n\sqrt{\ln n}}{n\sqrt{\ln n} + \sqrt{n}\ln n} = \lim_{n\to\infty} \frac{1}{1 + \sqrt{\frac{\ln n}{n}}} = 1$$

since $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ (this is justified by L'Hospital's rule). Therefore, $\sum \frac{1}{n\sqrt{\ln n} + \sqrt{n}\ln n}$ converges if and only if $\sum \frac{1}{n\sqrt{\ln n}}$ converges.

Now note that $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ diverges by the integral test since the improper integral

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{\ln 2}^{\infty} \frac{1}{\sqrt{u}}\,du \qquad \text{(substitution with } u = \ln x\text{)}$$

diverges. So $\sum \frac{1}{n\sqrt{\ln n}}$ diverges and the original series is *not* absolutely convergent.

[There are many other ways to solve this problem by comparing against a series which we know is divergent. For example, since $\frac{1}{n\sqrt{\ln n}+\sqrt{n}\ln n} \geq \frac{1}{2n\ln n}$ and $\sum \frac{1}{2n\ln n}$ diverges by the integral test, we find that the original series is not absolutely convergent.]

b) If not, is the series conditionally convergent or divergent? Justify your use of any test.

Solution: We use the alternating series test. Let

$$b_n = \frac{1}{n\sqrt{\ln n}+\sqrt{n}\ln n}.$$

First note that $b_n \to 0$ as $n \to \infty$. Next, since n and $\ln n$ are increasing functions of n we must have that $b_{n+1} \leq b_n$ for all n. The alternating series test then implies that the series is convergent. Since it is not absolutely convergent, it is conditionally convergent.

Question #2 (20 points)

Consider the power series

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}.$$

a) Find R, the radius of convergence of the series.

Solution: We will use the ratio test. Let

$$a_n = \frac{(3x-2)^n}{n3^n}.$$

The series converges when

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(3x-2)^n}\cdot\frac{n3^n}{(n+1)3^{n+1}}\right| \\
&= \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{|3x-2|}{3} \\
&= \left|x-\frac{2}{3}\right| < 1
\end{aligned}$$

and diverges when $\left|x-\frac{2}{3}\right| > 1$. Therefore, the radius of convergence is $R=1$.

b) Find I, the interval of convergence of the series. At the least, name any tests you use.

Solution: The interval of convergence is necessarily either $\left(-\frac{1}{3},\frac{5}{3}\right)$, $\left(-\frac{1}{3},\frac{5}{3}\right]$, $\left[-\frac{1}{3},\frac{5}{3}\right)$, or $\left[-\frac{1}{3},\frac{5}{3}\right]$. For $x=-\frac{1}{3}$ the series is

$$\sum_{n=1}^{\infty}\frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}.$$

It is the alternating harmonic series, which we know converges (this can be shown by using the alternating series test). For $x=\frac{5}{3}$, the series is

$$\sum_{n=1}^{\infty}\frac{3^n}{n3^n} = \sum_{n=1}^{\infty}\frac{1}{n},$$

which diverges as it is the harmonic series (i.e., a p-series with $p = 1$). To conclude, the interval of convergence is $I = \left[-\frac{1}{3}, \frac{5}{3}\right)$.

Question #3 (25 points)

Consider the function

$$f(x) = \ln\left(1 + \frac{1}{2}x^2\right).$$

a) Find its second-degree Taylor polynomial $T_2(x)$ **centered at** $a = 2$.

Solution: The polynomial $T_2(x)$ centered at 2 necessarily has the form

$$T_2(x) = f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2.$$

Since

$$f'(x) = \frac{x}{1 + \frac{1}{2}x^2}, \qquad f''(x) = \frac{1 - \frac{1}{2}x^2}{\left(1 + \frac{1}{2}x^2\right)^2}$$

we get that $f(2) = \ln 3$, $f'(2) = 2/3$, $f''(2) = -1/9$. So

$$T_2(x) = \ln(3) + \frac{2}{3}(x-2) - \frac{1}{18}(x-2)^2.$$

b) Find the Maclaurin series of f (i.e., series centered at $a = 0$) **and** determine its radius of convergence. [Hint: Try using the known power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

to find the answer quickly.]

Solution: First note that

$$f'(x) = \frac{x}{1 + \frac{1}{2}x^2} = x \cdot \frac{1}{1 - \left(-\frac{1}{2}x^2\right)}.$$

Substituting $-\frac{1}{2}x^2$ for x in the series expansion of $\frac{1}{1-x}$ gives that

$$x \cdot \frac{1}{1 - \left(-\frac{1}{2}x^2\right)} = x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n}$$

when $\left|-\frac{1}{2}x^2\right| < 1$, i.e., when $|x| < \sqrt{2}$. We integrate the power series term by term to obtain

$$f(x) \;=\; \int \left(x \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^{2n}\right) dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n}{2^n} x^{2n+1} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(2n+2)} x^{2n+2}.$$

To find the constant of integration C, note that $C = f(0) = \ln 1 = 0$. To conclude,

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(2n+2)} x^{2n+2}, \qquad R = \sqrt{2}.$$

This print-out should have 4 questions. Multiple-choice questions may continue on the next column or page – find all choices before answering.

CalC12g01a
001 10.0 points

Which, if any, of the following statements are true?

A. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.

B. The Ratio Test can be used to determine whether $\sum 1/n^3$ converges.

C. If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent.

1. C only **correct**

2. A only

3. all of them

4. A and B only

5. B only

6. B and C only

7. none of them

8. A and C only

Explanation:

A. False: when $a_n = 1/n$, then $\lim_{n\to\infty} a_n = 0$, but

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges by the Integral Test.

B. False: when $a_n = 1/n^3$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^3}{(n+1)^3} \longrightarrow 1$$

as $n \to, \infty$, so the Ratio Test is inconclusive.

C. True: if $\sum |a_n|$ were convergent, then $\sum a_n$ would be absolutely convergent, hence convergent.

CalC12f33c
002 10.0 points

Determine which, if any, of the following series converge.

$$(A) \quad \sum_{n=1}^{\infty} \frac{6^n}{(n+6)^n}$$

$$(B) \quad \sum_{n=1}^{\infty} \left(\frac{3n}{6n+7}\right)^n \left(\frac{6}{5}\right)^n$$

$$(C) \quad \sum_{n=1}^{\infty} n! \left(\frac{3}{n}\right)^n$$

1. C only

2. A only

3. none of them

4. B only

5. B and C

6. all of them

7. A and C

8. A and B **correct**

Explanation:

To check for convergence we shall use either the Ratio test or the Root test which means computing one or other of

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, \qquad \lim_{n\to\infty} |a_n|^{1/n}$$

for each of the given series.

(A) The root test is the better one to use because

$$|a_n|^{1/n} = \frac{6}{n+6} \longrightarrow 0$$

as $n \to \infty$, so series (A) converges.

(B) The root test again is the better one to use because

$$|a_n|^{1/n} = \frac{6}{5}\left(\frac{3n}{6n+7}\right) \longrightarrow \frac{3}{5} < 1$$

as $n \to \infty$, so series (B) converges.

(C) The ratio test is the better one to use because

$$\left|\frac{a_{n+1}}{a_n}\right| = 3\left(\frac{(n+1)!n^n}{n!(n+1)^{n+1}}\right).$$

But

$$\frac{(n+1)!}{n!} = n+1\,,$$

while

$$\frac{n^n}{(n+1)^{n+1}} = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n.$$

Thus

$$\frac{a_{n+1}}{a_n} = 3\left(\frac{n}{n+1}\right)^n \longrightarrow \frac{3}{e} > 1$$

as $n \to \infty$, so series (C) diverges.

Consequently, of the given infinite series,

$$\boxed{\text{only } A \text{ and } B}$$

converge.

CalC7g23d
003 10.0 points

Find the value of

$$\lim_{x\to\infty} \frac{x^3}{3^x}.$$

1. limit $= 0$ **correct**

2. limit $= 3$

3. none of the other answers

4. limit $= \dfrac{1}{3}$

5. limit $= \infty$

6. limit $= -\infty$

Explanation:

Set

$$f(x) = x^3, \quad g(x) = 3^x = e^{x\ln 3}.$$

Then f, g are everywhere differentiable functions such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$$

Thus L'Hospital's Rule applies, in which case

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

Now

$$f'(x) = 3x^2\,, \qquad g'(x) = (\ln 3)\,3^x.$$

But then

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{3x^2}{(\ln 3)\,3^x}\,,$$

which, up to a constant, is the same limit we started with except that x^3 in the numerator has become x^2. Consequently, if we apply L'Hospital's rule sufficiently often, we finally end up with

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\text{constant}}{3^x}.$$

Thus

$$\boxed{\lim_{x\to\infty} \frac{x^3}{3^x} = 0}.$$

CalC12b50s
004 10.0 points

If the n^{th} partial sum S_n of an infinite series

$$\sum_{n=1}^{\infty} a_n$$

is given by

$$S_n \;=\; 3 - \frac{n}{2^n}\,,$$

find a_n for $n > 1$.

1. $a_n \;=\; \dfrac{n-2}{2^n}$ **correct**

2. $a_n \;=\; \dfrac{n-2}{2^{n-1}}$

3. $a_n \;=\; 3\left(\dfrac{n-2}{2^n}\right)$

4. $a_n \;=\; \dfrac{3n-2}{2^n}$

5. $a_n \;=\; 3\left(\dfrac{3n-2}{2^n}\right)$

6. $a_n \;=\; 3\left(\dfrac{n-2}{2^{n-1}}\right)$

Explanation:

By definition, the n^{th} partial sum of

$$\sum_{n=1}^{\infty} a_n$$

is given by

$$S_n \;=\; a_1 + a_2 + \cdots + a_n\,.$$

In particular,

$$a_n \;=\; \begin{cases} S_n - S_{n-1}, & n > 1, \\ S_n, & n = 1. \end{cases}$$

Thus

$$\begin{aligned} a_n \;=\; S_n - S_{n-1} \;&=\; \frac{n-1}{2^{n-1}} - \frac{n}{2^n} \\ &=\; \frac{2\,(n-1)}{2^n} - \frac{n}{2^n} \end{aligned}$$

when $n > 1$. Consequently,

$$\boxed{a_n \;=\; \frac{n-2}{2^n}}$$

for $n > 1$.