M408D (54690/95/00), Quiz #1 Solutions, 09/08/2010

Question #1

Is the improper integral

$$\int_1^\infty \frac{\ln x}{x^2}\,dx$$

convergent? If so, what is its value? [Hint: Use integration by parts, which says that $\int_a^b f(x)h'(x)dx = f(x)h(x)\Big|_{x=a}^{x=b} - \int_a^b f'(x)h(x)dx$.]

Solution: The integral is convergent. We show this as follows. First, by definition,

$$\int_1^\infty \frac{\ln x}{x^2}\,dx \;=\; \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx.$$

Now using integration by parts with $f(x)=\ln x$ and $h'(x)=1/x^2$ (and therefore, $f'(x)=1/x$ and $h(x)=-1/x$) we have that

$$\begin{aligned}\int_1^t \frac{\ln x}{x^2}\,dx &= -\frac{\ln x}{x}\Big|_{x=1}^{x=t} + \int_1^t \frac{1}{x^2}\,dx \\ &= -\frac{\ln t}{t} + \left(1-\frac{1}{t}\right).\end{aligned}$$

Taking the limit as $t\to\infty$, we see that the first term on the last line is of indeterminate form ∞/∞ and the second term goes to 1. For the first term, we use L'Hospital's rule:

$$\lim_{t\to\infty}\frac{\ln t}{t} = \lim_{t\to\infty}\frac{1/t}{1} = 0.$$

Therefore,

$$\lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = 1$$

and the improper integral is convergent with value 1.

Question #2

Consider the sequence $\{a_n\}$ given by

$$a_n = \left(1+7n^3\right)^{1/n}.$$

a) Is $\{a_n\}_{n=1}^\infty$ a convergent sequence? If so, what is its limit?

Solution: Let $f(x) = \left(1+7x^3\right)^{1/x}$ so that f is a continuous function which interpolates a_n (i.e., $f(n)=a_n$). Then we know that if $\lim_{x\to\infty} f(x)$ exists and has a finite limit, then the same must be true for $\lim_{n\to\infty} a_n$ with the same limit. Since

$$\lim_{x\to\infty}\left(1+7x^3\right)^{1/x} = \lim_{x\to\infty}\exp\left(\frac{1}{x}\ln\left(1+7x^3\right)\right) = \exp\left(\lim_{x\to\infty}\frac{\ln\left(1+7x^3\right)}{x}\right),$$

we must use L'Hospital's rule since we have an indeterminate limit of the form ∞/∞. Therefore,

$$\lim_{x\to\infty} \frac{\ln\left(1+7x^3\right)}{x} = \lim_{x\to\infty} \frac{\frac{21x^2}{1+7x^3}}{1} = 0$$

and

$$\lim_{x\to\infty} \left(1+7x^3\right)^{1/x} = e^0 = 1,$$

i.e., the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent with limit 1.

b) Now consider the infinite series $\sum_{n=1}^{\infty} a_n$. Is this a convergent series? Why or why not?

Solution: No, the series is divergent since $\lim_{n\to 0} a_n \neq 0$ by part (a). So by the divergence test, the series must diverge!