M408D (54690/95/00), Quiz #2, 09/22/2010

**Question #1**

Define the series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}+2n\ln n}.$$

a) Is the series absolutely convergent? Justify your answer and any tests you use.

**Solution:** The series is not absolutely convergent. To see this, we show that

$$\sum_{n=2}^{\infty} \left|\frac{(-1)^{n-1}}{\sqrt{n}+2n\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+2n\ln n}$$

diverges. Note that we are working with a series with only positive terms. To simplify matters, we use the limit comparison test to compare this series to

$$\sum_{n=2}^{\infty} \frac{1}{2n\ln n}.$$

Let $c_n = \frac{1}{\sqrt{n}+2n\ln n}$ and $d_n = \frac{1}{2n\ln n}$. Since

$$\lim_{n\to\infty} \frac{c_n}{d_n} = \lim_{n\to\infty} \frac{2n\ln n}{\sqrt{n}+2n\ln n} = \lim_{n\to\infty} \frac{1}{\frac{1}{2\sqrt{n}\ln n}+1} = 1,$$

the limit comparison test holds and $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+2n\ln n}$ diverges if $\sum_{n=2}^{\infty} \frac{1}{2n\ln n}$ diverges. Finally, we know that $\sum_{n=2}^{\infty} \frac{1}{2n\ln n}$ diverges by the integral test since the improper integral

$$\int_2^{\infty} \frac{1}{2x\ln x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{2u}\,du$$

diverges. Therefore, the original series is not absolutely convergent.

b) If not, is the series conditionally convergent or divergent? Justify your use of any test.

**Solution:** The series is conditionally convergent. To see this, we use the alternating series test. Here, the terms of the series are of the form

$$a_n = (-1)^{n-1} b_n, \qquad b_n = \frac{1}{\sqrt{n}+2n\ln n}.$$

First we note that $\lim_{n\to\infty} b_n = 0$. Next, since $\sqrt{n} + 2n \ln n$ is increasing in $n$, we have that $\frac{1}{\sqrt{n}+2n\ln n}$ is decreasing in $n$. That is, $b_{n+1} \le b_n$ for all $n$. The alternating series test therefore holds and the series is conditionally convergent.

**Question #2**

Consider the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}(x-5)^n.$$

a) Find $R$, the radius of convergence of the series.

**Solution:** We use the ratio test to find $R$. With $a_n = \frac{(-1)^n}{n^{1/3}}(x-5)^n$, we have that

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{(n+1)^{1/3}}\cdot\frac{n^{1/3}}{(x-5)^n}\right| \\ &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/3}|x-5| \\ &= |x-5|. \end{aligned}$$

The test implies that the series converges when $|x-5|<1$ and diverges when $|x-5|>1$. Therefore, $R=1$.

b) Find $I$, the interval of convergence of the series. At the least, name any tests you use.

We know that the series converges at least when $|x-5|<1$, i.e. when $4<x<6$. Now we must test the boundaries of the interval. In the case $x=4$, the series diverges by the $p$-series test with $p=1/3$ since

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}(-1)^n = \sum_{n=1}^{\infty}\frac{1}{n^{1/3}}.$$

In the case $x=6$, the series converges by the alternating series test since

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/3}}(1)^n = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{1/3}},$$

and with $b_n = \frac{1}{n^{1/3}}$ we find that $\lim_{n\to\infty} b_n = 0$ and $b_{n+1}\le b_n$ for all $n$. To conclude, the interval of convergence is $I=(4,6]$.