M408D (54690/95/00), Quiz #3 Solutions

Question #1

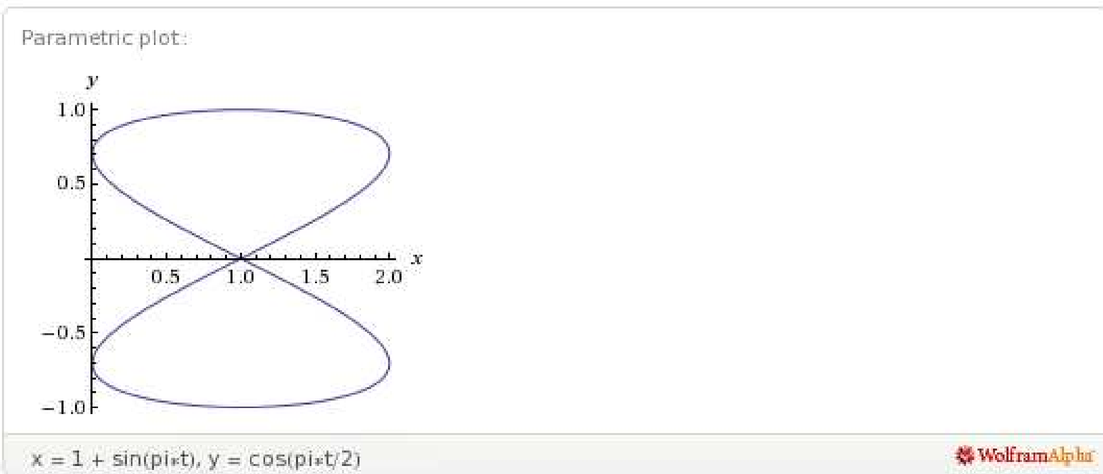
Consider the "hourglass shape" given by the parametric equations

$$x = 1 + \sin(\pi t), \quad y = \cos(\pi t/2)$$

with $-2 \leq t < 2$.

a) The curve intersects itself once at the point (1, 0). Find the slope of one of the two tangent lines to the curve at $(1, 0)$ using the formula $dy/dx = \frac{dy/dt}{dx/dt}$.

Solution: Drawing the parametric curve yields the figure:



The slope of the tangent line at a point $(x(t), y(t))$ on the curve is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{\sin(\pi t/2)}{2\cos(\pi t)}.$$

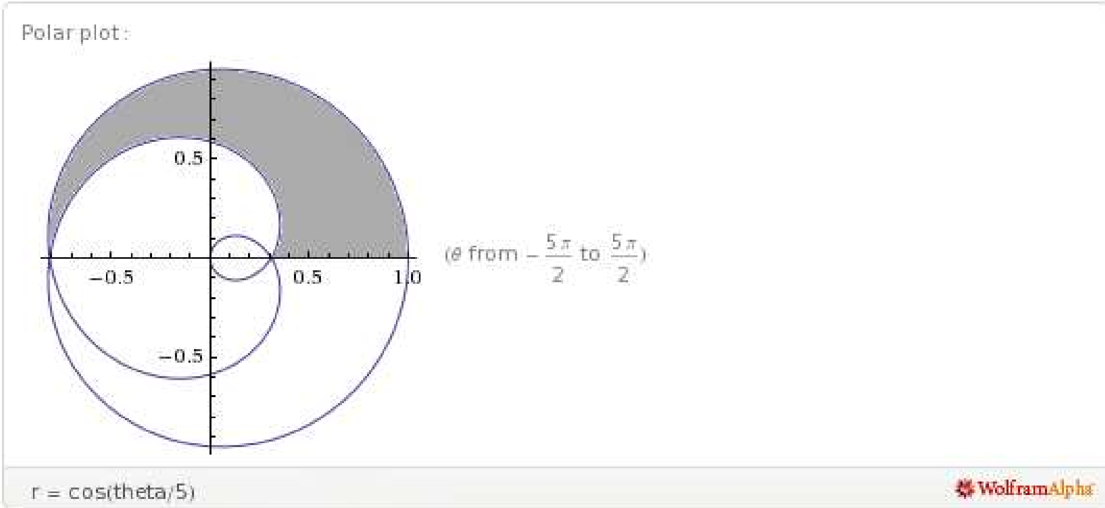
Since the point $(1, 0)$ is reached at $t = -1$ and $t = 1$, this gives that the slopes of the two tangent lines at $(1, 0)$ are $-1/2$ and $1/2$, respectively.

b) Find all (x, y) such that the curve has a horizontal tangent.

Solution: We seek all $(x(t), y(t))$ such that $dy/dt = 0$ while $dx/dt \neq 0$. First note that $dy/dt = -\sin(\pi t/2) = 0$ when $t = -2$ or $t = 0$ (since we restrict ourselves to $-2 \leq t < 2$). It is easy to check that for $dx/dt = 2\cos(\pi t) \neq 0$ at these values of t. Therefore, the curve has horizontal tangents at $(x(-2), y(-2))$ and $(x(0), y(0))$, i.e., at $(1, -1)$ and $(1, 1)$.

Question #2

Consider the polar equation $r = \cos(\theta/5)$, graphed below.



a) Give an *expression* for the area of the shaded region (i.e., only write down the proper integral(s), don't evaluate them!) using the polar area formula $A = \int_a^b \frac{1}{2} r^2 d\theta$. Make sure that you specify correct boundaries of integration for any integral(s).

Solution: We only need to find the proper boundaries of integration. Note that when $\theta = 0$ we start at the point $(1,0)$ (in Cartesian coordinates), then trace out the outermost part of the curve as we increase θ, passing through the x-axis again when $\theta = \pi$. Using the symmetry of the curve we find that the shaded area is

$$A = \int_0^{\pi} \frac{1}{2}\cos^2(\theta/5)\,d\theta - \int_{\pi}^{2\pi} \frac{1}{2}\cos^2(\theta/5)\,d\theta.$$

There are many other ways to express this area, such as

$$A = \int_0^{\pi} \frac{1}{2}\cos^2(\theta/5)\,d\theta + \int_{-\pi}^{-2\pi} \frac{1}{2}\cos^2(\theta/5)\,d\theta.$$

b) Give an *expression* for the arc length of the part of the curve that lies to the right of the y-axis (again, no need to evaluate) using the polar arc length formula $L_C = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$. Be careful when specifying boundaries of integration.

Solution:

$$L_C = 2\left(\int_0^{\pi/2} \sqrt{\cos^2(\theta/5) + \frac{1}{25}\sin^2(\theta/5)}\,d\theta + \int_{3\pi/2}^{5\pi/2} \sqrt{\cos^2(\theta/5) + \frac{1}{25}\sin^2(\theta/5)}\,d\theta\right).$$

Again, there are many other ways to express this quantity.