M408D (54690/95/00), Quiz #4 Solutions

Question #1

Define the vectors

$$\boldsymbol{u} = \langle 1, 0, -2 \rangle$$
$$\boldsymbol{v} = \langle 4, 5, 2 \rangle$$
$$\boldsymbol{w} = \langle 1, 2, 1 \rangle.$$

a) Using the scalar triple product $V = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$ find the volume of the parallelepiped determined by $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$.

Solution: We first find that

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & 5 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} = \langle 1, -2, 3 \rangle.$$

Then $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = 1 - 6 = -5$ so the volume of the parallelepiped is $V = 5$.

b) Suppose we consider a plane A which is parallel to the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, and passes through the point $\boldsymbol{u}$. Determine the scalar equation for the plane (which takes the form $a(x - x_0) + b(y - y_0) + c(z - z_0) + d = 0$ with constants $a, b, c, d, x_0, y_0, z_0$).

[Hint: Find a normal vector $\boldsymbol{n}$ to the plane. Then use the vector equation $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$ for a plane with this normal vector which passes through a given point $\boldsymbol{r_0}$.]

Solution: Since the plane A is parallel to $\boldsymbol{v}$ and $\boldsymbol{w}$ and $\boldsymbol{v} \times \boldsymbol{w}$ is orthogonal to both vectors, we find that a normal vector to A is $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = \langle 1, -2, 3 \rangle$ by part (a). Here, $\boldsymbol{r_0} = \boldsymbol{u}$. Therefore, the vector equation for the plane is

$$\langle 1, -2, 3 \rangle \cdot (\langle x, y, z \rangle - \langle 1, 0, -2 \rangle) = 0.$$

Expanding this out gives the desired answer:

$$(x - 1) - 2y + 3(z + 2) = 0 \qquad \text{or} \qquad x - 2y + 3z + 5 = 0.$$

c) (Bonus: +1 point) Suppose we have a plane B with normal vector $\boldsymbol{m}$. The orthogonal projection $\mathbf{proj}_B\boldsymbol{u}$ of $\boldsymbol{u}$ onto the plane B is the vector projection of $\boldsymbol{u}$ onto the line obtained by intersecting B with a plane C parallel to $\boldsymbol{u}$ and orthogonal to B (see diagram). Give an expression for $\mathbf{proj}_B\boldsymbol{u}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{m}$.

[Hint: Remember that $\mathbf{proj}_{\boldsymbol{b}}\boldsymbol{a} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\right)\boldsymbol{b}$.]

Solution: The easiest way to derive the expression for $\mathbf{proj}_B\boldsymbol{u}$ is to note that it is the difference between $\boldsymbol{u}$ and the vector projection $\mathbf{proj}_{\boldsymbol{m}}\boldsymbol{u}$ of $\boldsymbol{u}$ onto $\boldsymbol{m}$:

$$\boldsymbol{u} = \mathbf{proj}_{\boldsymbol{m}}\boldsymbol{u} + \mathbf{proj}_B\boldsymbol{u}.$$

Therefore, $\mathbf{proj}_B\boldsymbol{u} = \boldsymbol{u} - \mathbf{proj}_{\boldsymbol{m}}\boldsymbol{u} = \boldsymbol{u} - \left(\frac{\boldsymbol{u} \cdot \boldsymbol{m}}{|\boldsymbol{m}|^2}\right)\boldsymbol{m}$.

Question #2

Consider the quadric surface given by

$$x^2 + y^2 = 9z^2.$$

a) Find the trace of the surface in the plane $-y + 3z = 2$. What kind of curve is this?

Solution: The trace is the intersection of the surface with the plane, and so must satisfy both equations. Therefore, we substitute $3z = y + 2$ into the first equation to obtain that $x^2 + y^2 = (y+2)^2 = y^2 + 4y + 4$, i.e., that

$$y = \frac{x^2}{4} - 1$$

which is the equation for a parabola. Alternatively, this can be written in terms of z instead of y as

$$z = \frac{x^2}{12} + \frac{1}{3}.$$

b) Is the surface $x^2 + y^2 = 9z^2$ an ellipsoid or a cone? Why?

[Hint: Remember that traces of ellipsoids are always ellipses, while traces of cones can be any conic section—ellipses, parabolas, or hyperbolas.]

Solution: The surface cannot be an ellipsoid since we see from part (a) that we can obtain traces of this surface that are not ellipses (in this case, a parabola). Therefore, it must be a cone. This can also be checked from the form of the equation or by considering vertical and horizontal traces.