M408D (54690/95/00), Sample Final Exam

Part 1:

Multiple choice questions (5 points each)

Solutions: 4, 5, 5, 5.

Question 1 (25 points)

Define

$$h(x,y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

a) Find all critical points of h.

   Solution: $(0,0)$, $(-5/3,0)$, $(-1,\pm 2)$.

b) Classify all critical points (i.e., determine if they are local maxima, minima, or saddle points) by using the second derivative test. To begin, compute the Hessian of $h(x,y)$.

   Solution: $(0,0)$ local minimum, $(-5/3,0)$ local maximum, $(-1,\pm 2)$ saddle points.

c) What are the absolute maximum and minimum values of h on the domain $D_1 = \{(x,\, y)\colon 0 \le x \le 4, 0 \le y \le 5\}$?

   Solution: $h(4,\, 5) = 433$ is the absolute maximum value, while $h(0,\, 0) = 0$ is the absolute minimum.

d) Using the method of Lagrange multipliers ($\nabla f = \lambda \nabla g$, $g = 0$ with constraint function g), determine the extreme values of the function

$$f(x,y) = xy^2$$

   on the ellipse $x^2 + \frac{1}{4}y^2 = 1$.

   Solution: Max. value $= \frac{8}{3\sqrt{3}}$, min. value $= \frac{-8}{3\sqrt{3}}$

Question #2 (20 points)

Evaluate the following integrals. Remember that iterated integrals are sometimes easier to evaluate after switching the order of integration, or after changing to different coordinates.

a) $\int_1^4 \int_1^2 \left(\frac{x}{y} - \frac{y}{x}\right) dy\, dx$

   Solution: $\frac{9}{2}\ln 2$

b) $\int_0^1 \int_{3y}^3 e^{x^2} dx\, dy$

   Solution: $\frac{e^9 - 1}{6}$

c) $\int_0^1 \int_y^{\sqrt{2-y^2}} (x+y)\, dx\, dy$

Solution: $\frac{2\sqrt{2}}{3}$

Question #3 (20 points)

Consider the linear transformation $T\colon (u,v) \to (x,y)$ given by

$$x = \frac{1}{2}(u-v)$$
$$y = \frac{1}{2}(u+v) \quad .$$

a) Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ of the transformation.

   Solution: $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}$

b) Determine the region in the uv-plane which maps to the trapezoidal region R in the xy-plane with vertices $(1,0)$, $(2,0)$, $(0,-2)$, $(0,-1)$.

   Solution: The trapezoid in the uv-plane with vertices $(1,-1)$, $(2,-2)$, $(-2,-2)$, $(-1,-1)$.

c) Use parts (a) and (b) to evaluate the integral

$$\iint_R e^{(x+y)/(x-y)} dA$$

   where R is the trapezoidal region of the xy-plane defined in part (b).

   Solution: $\frac{3}{4}(e - e^{-1})$.

Question #4 (15 points)

Define the function

$$f(x,y,z) = x\sin(yz).$$

a) Determine $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$.

   Solution: $\frac{\partial f}{\partial x} = \sin(yz)$, $\frac{\partial f}{\partial y} = xz\cos(yz)$, $\frac{\partial f}{\partial z} = xy\cos(yz)$.

b) Find the directional derivative $D_{\boldsymbol{u}}f$ of f at the point $P(2,\ 0,\ 1)$ in the direction of the vector $\boldsymbol{v} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$.

   Solution: $D_{\boldsymbol{u}}f|_{(2,0,1)} = \langle \sin(yz),\ xz\cos(yz),\ xy\cos(yz)\rangle\Big|_{(2,0,1)} \cdot \frac{1}{\sqrt{6}}\langle 1,-1,2\rangle = -\frac{2}{\sqrt{6}}$.

c) Suppose

$$x(s,t) = \cos(s^2+t)$$
$$y(s,t) = e^{-2st}$$
$$z(s,t) = s^3 - 2st^2 + 4.$$

   Use the chain rule to determine $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

   Solution:

   $\frac{\partial f}{\partial s} = -2s\sin(s^2+t)\sin(yz) - 2te^{-2st}xz\cos(yz) + (3s^2 - 2t^2)xy\cos(yz)$

$\frac{\partial f}{\partial t} = -\sin(s^2+t)\sin(yz) - 2se^{-2st}xz\cos(yz) - 4sxy\cos(yz)$

Part 2:

Multiple choice questions (5 points each)

Solutions: 4, 3, 1, 2.

Question 1 (25 points)

Determine whether the following series are convergent or divergent. Justify your use of any test.

a) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

Solution: Convergent by comparison test with convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

b) $\sum_{n=6}^{\infty} (-1)^n \frac{3+2n+4n^2}{n^2+n+\sin(n)}$

Solution: Divergent by divergence test (since $\lim_{n\to\infty} |a_n| = 4$).

c) $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

Solution: Convergent by ratio test.

d) $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

Solution: Convergent by root test.

Question #2 (20 points)

Find the Taylor series of the following functions about the given point a (using any method) and find the corresponding interval of convergence of the series.

a) $f(x) = 2x\cos(x^2)$; $a = 0$

Solution: $f(x) = 2x\left(\sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{2x^{4n+1}}{(2n)!}$, $I = (-\infty, \infty)$

b) $f(x) = x^{-2}$; $a = 1$

Solution: $f(x) = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$, $I = (0, 2)$

Question #3 (20 points)

Define the vector-valued function

$$\boldsymbol{r}(t) = \langle 2e^t, 3t^2, te^{4t} \rangle.$$

a) What is $\boldsymbol{r}''(t)$?

Solution: $\boldsymbol{r}''(t) = \langle 2e^t, 6, 8e^{4t} + 16te^{4t} \rangle$

b) Write an integral in t for the arc length of the curve between the points $P(2,\ 0,\ 0)$ and $Q(2e, 3, e^4)$. Do not attempt to evaluate the integral.

Solution: $L = \int_0^1 |\langle 2e^t, 6t, e^{4t}(1+4t)\rangle| dt = \int_0^1 \sqrt{4e^{2t} + 36t^2 + e^{8t}(1+4t)^2} dt.$

c) At what point $P(x_0, y_0, z_0)$ does the curve $\boldsymbol{r}(t)$ intersect the surface $16z = x^4$?

Solution: When $t = 1$, i.e., the point $P(2e, 3, e^4)$.

d) Find the unit tangent vector $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$ at the point $P(2, 0, 0)$.

Solution: When $t = 0$, the curve is at $P(2, 0, 0)$. The corresponding tangent vector is

$$\boldsymbol{T}(0) = \frac{\langle 2e^t, 6t, e^{4t}(1+4t)\rangle}{4e^{2t} + 36t^2 + e^{8t}(1+4t)^2}\bigg|_{t=0} = \frac{1}{5}\langle 2, 0, 1\rangle.$$

Question #4 (15 points)

Define the vectors

$$\boldsymbol{a} = \langle 4, 2, 0\rangle$$
$$\boldsymbol{b} = \langle 1, -3, 5\rangle$$
$$\boldsymbol{c} = \langle -2, 2, 1\rangle.$$

a) Find $\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a})$.

Solution: 74

b) What is the cosine of the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{c}$?

Solution: $-\frac{1}{\sqrt{13}}$

c) Determine the equation for the plane parallel to the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that passes through the point $P(1, 1, 1)$ (write in the form $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r_0}) = 0$).

Solution: Take $\boldsymbol{n} = \boldsymbol{a} \times \boldsymbol{b} = \langle 10, -20, -14\rangle$ and $\boldsymbol{r_0} = \langle 1, 1, 1\rangle$.