M408D (54690/95/00), Sample Midterm #2 Solutions

Multiple choice questions (5 points each)

Solutions: 5, 3, 2, 2, 2.

Question #1 (20 points)

Define

$$\boldsymbol{u} = \langle 0, -1, 2\rangle$$
$$\boldsymbol{v} = \langle 3, 4, 5\rangle$$
$$\boldsymbol{w} = \langle -3, 7, 1\rangle.$$

a) What is $\boldsymbol{u} \times \boldsymbol{v} \times \boldsymbol{w}$?

   Solution: $\boldsymbol{u} \times \boldsymbol{v} \times \boldsymbol{w} = (-15, 4, -73)$.

b) Find a vector equation for the plane parallel to $\boldsymbol{v}$ and $\boldsymbol{w}$ that passes through the point $\boldsymbol{u}$.

   Solution: $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{u}) = 0$ where $\boldsymbol{n} = \boldsymbol{v} \times \boldsymbol{w} = (-31, -18, 33)$.

c) What is the scalar projection $\text{comp}_{\boldsymbol{w}}\boldsymbol{u}$? [Hint: Remember that the scalar projection of $\boldsymbol{a}$ onto a unit vector $\boldsymbol{e}$ is $|\boldsymbol{a} \cdot \boldsymbol{e}|$. Now take $\boldsymbol{e} = \boldsymbol{b}/|\boldsymbol{b}|$ to find the scalar projection onto some arbitrary vector $\boldsymbol{b}$.]

   Solution: $\text{comp}_{\boldsymbol{w}}\boldsymbol{u} = \frac{1}{\sqrt{59}}(0, -7, 2)$.

Question #2 (20 points)

Consider the quadric surface given by

$$6z = x^2 + 4y^2 - 1.$$

a) Find the trace of the surface in the plane $2y - z - 1 = 0$. What kind of curve is this?

   Solution: $x^2 + (z-2)^2 = 4$, circle with radius 2.

b) Does the curve $\boldsymbol{r}(t) = \langle \sin t, \frac{1}{2}\cos t, t\rangle$ lie on the surface $6z = x^2 + 4y^2 - 1$? If not, at what point $P(x_0, y_0, z_0)$ does it intersect it?

   Solution: No, but it does intersect it when $t = 0$, i.e., at the point $P(0, 1/2, 0)$.

Question #3 (20 points)

Define

$$\boldsymbol{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

and suppose $\boldsymbol{r}(0) = \langle 1, 2, 3\rangle$.

a) What is $\boldsymbol{r}(t)$?

   Solution: $\boldsymbol{r}(t) = (-1 + 2\cos t, 2 + 2\sin t, 3)$.

b) Find the unit tangent vector $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$.

Solution: $\boldsymbol{T}(t) = \langle -\sin t, \cos t, 0 \rangle$.

c) What is the curvature $\kappa(t) = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|}$ of the curve at the point $P(1, 2, 3)$?

Solution: $\boldsymbol{T}'(t) = \langle -\cos t,\ -\sin t,\ 0 \rangle$. The point $P(1,\ 2,\ 3)$ corresponds to $t = 0$, so since $\boldsymbol{T}'(0) = \langle -1, 0, 0 \rangle$ and $|\boldsymbol{r}'(0)| = 2$, $\kappa(0) = 1/2$.

d) Determine a normal vector to the curve at $P(1, 2, 3)$. [Hint: Remember that $\boldsymbol{T}$ and $\boldsymbol{T}'$ are perpendicular to each other!]

Solution: A normal vector is given by $\boldsymbol{T}'(0) = \langle -1, 0, 0 \rangle$.

Question #4 (15 points)

Consider the polar equation

$$r = \sin\theta.$$

a) Rewrite this equation in Cartesian coordinates and graph the curve. That is, find the corresponding equation in x and y. What conic section is this?

Solution: $x^2 + y^2 = y$ or $x^2 + (y - 1/2)^2 = 1/4$, a circle with radius $1/2$.

b) The polar curve $r = 2\theta$ lies further away from the origin than $r = \sin\theta$ since for every θ $2\theta \geq \sin\theta$. Find the area that lies between these two curves between the angles $\theta = 0$ and $\theta = \pi/2$ using the formula $A = \frac{1}{2}\int_{\theta_0}^{\theta_1} |f^2(\theta) - g^2(\theta)|\, d\theta$ for the area between two polar functions.

Solution: $A = \frac{1}{2}\int_0^{\pi/2} (4\theta^2 - \sin^2(\theta))d\theta$.