M427L (55200), Final exam solutions

Question #1.1 (30 points)

Suppose

$$\boldsymbol{F}(x,y,z) = -z^2\boldsymbol{j} + yz\boldsymbol{k}.$$

a) Does there exist a vector field $\boldsymbol{G}$ such that $\nabla \times \boldsymbol{G} = \boldsymbol{F}$? Explain why or why not.

Solution: Since $\nabla \cdot \boldsymbol{F} = y \neq 0$, then there cannot exist a $\boldsymbol{G}$ such that $\nabla \times \boldsymbol{G} = \boldsymbol{F}$ because the divergence of a curl is always 0.

b) Is $\boldsymbol{F}$ a gradient vector field? Justify your answer.

Solution: No, since

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -z^2 & yz \end{vmatrix} = 3z\boldsymbol{i} \neq \boldsymbol{0}$$

but the curl of a gradient is always $\boldsymbol{0}$.

c) Let $S = \{(x,y,z) : x^2+y^2+z^2 = 1, x \geq 0\}$ be the right half of the unit sphere. Compute the surface integral

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$$

without using any integral theorems by choosing a convenient parametrization of S.

Solution: Let $x = \sin\phi\cos\theta,\ y = \sin\phi\sin\theta,\ z = \cos\phi$ with $-\pi/2 \leq \theta \leq \pi/2,\ 0 \leq \phi \leq \pi$. Then

$$\begin{aligned} \iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} &= \iint_S 3z\boldsymbol{i} \cdot (x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) dS \\ &= 3\int_0^{\pi} \int_{-\pi/2}^{\pi/2} \sin^2\phi\cos\phi\cos\theta \, d\theta \, d\phi \\ &= 0. \end{aligned}$$

d) Verify your answer to (b) using Stokes' theorem and evaluating the appropriate integral. [Hint: Remember, $\sin^2 t + \cos^2 t = 1$.]

Solution: The boundary ∂S of S is the unit circle in the yz-plane, traversed in a counterclockwise manner. Parametrizing ∂S by $\mathbf{c}(t) = (0, \cos t, \sin t)$ with $0 \leq t \leq 2\pi$,

$$\begin{aligned} \iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} &= \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= \int_0^{2\pi} \left(-\sin^2 t\boldsymbol{j} + \cos t \sin t\boldsymbol{k}\right) \cdot (-\sin t\boldsymbol{j} + \cos t\boldsymbol{k}) dt \\ &= \int_0^{2\pi} \sin t\left(\sin^2 t + \cos^2 t\right) dt \\ &= 0. \end{aligned}$$

Question #1.2 (25 points)

Recall that Green's theorem relates the line integral of a 2-D vector field along a closed path to the integral of the scalar curl in the region enclosed by the path.

a) Suppose we have the line integral

$$\int_C \sin\left(x^3\right)dx + 2ye^{x^2}\,dy,$$

where C is the triangular path that connects the points $(0,0)$, $(2,2)$, and $(0,2)$ in a counterclockwise manner. Use Green's theorem to write this line integral as a double integral with the appropriate limits of integration. *Do not evaluate this integral yet.*

Solution: Let $\boldsymbol{F}(x,y) = P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$ with $P(x,y) = \sin\left(x^3\right)$ and $Q(x,y) = 2ye^{x^2}$. By Green's theorem

$$\int_C Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_0^2 \int_0^y 4xye^{x^2}dx\,dy.$$

b) Evaluate the double integral found in part (a).

Solution:

$$\int_0^2 \int_0^y 4xye^{x^2}dx\,dy = \int_0^2 2y\Big(e^{y^2} - 1\Big)dy = e^4 - 5.$$

c) [Note: This part is unrelated to parts (a) and (b).]

True or false? Since the vector field $\boldsymbol{F}(x,y) = \frac{1}{x^2+y^2}(-y\boldsymbol{i} + x\boldsymbol{j})$ has zero scalar curl at all points where it it is defined, Green's theorem implies that for the unit circle C

$$\int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 0.$$

Solution: False. The vector field is not continuously differentiable at the origin so we cannot apply Green's theorem to C. In fact, as we discussed in class this vector field has a point singularity at the origin—i.e., curl $\boldsymbol{F}(\boldsymbol{x}) = \delta_{\mathbf{0}}(\boldsymbol{x})$ where $\delta_{\mathbf{0}}(\boldsymbol{x})$ is the Dirac delta distribution at $\mathbf{0}$—so any line integral about the origin is nonzero.

Question #1.3 (25 points)

The velocity field of a fluid at a particular instant of time is given by

$$\boldsymbol{V}(x,y,z) = x^3\boldsymbol{i} + y^3\boldsymbol{j} + z\boldsymbol{k}.$$

Consider the fluid flux

$$\int\int_{\partial W} \boldsymbol{V}\cdot\boldsymbol{n}\,dS$$

through the surface ∂W of the closed cone $W = \{(x,y,z) : z^2 = x^2 + y^2, 0 \le z \le 1\}$. Evaluate this surface integral using the following steps:

a) Use the divergence theorem to express the flux through ∂W in terms of a triple integral (no need to write down boundaries of integration). *Do not evaluate this integral yet.*

Solution: Since $\nabla \cdot \boldsymbol{V} = 3x^2 + 3y^2 + 1$,

$$\int\int_{\partial W} \boldsymbol{V} \cdot \boldsymbol{n}\, dS = \int\int\int_W (\nabla \cdot \boldsymbol{V})\, dV = \int\int\int_W (3x^2 + 3y^2 + 1)\, dV.$$

b) Change to cylindrical coordinates ($x = r\cos\theta$, $y = r\sin\theta$, z unchanged) to evaluate the integral found in part (a). Make sure to include the Jacobian term $|\partial(x,y,z)/\partial(r,\theta,z)| = r\,dr\,d\theta\,dz$ in the transformed integral.

Solution:

$$\int\int\int_W (3x^2 + 3y^2 + 1)\, dV = \int_0^1 \int_0^{2\pi} \int_0^z (3r^2 + 1) r\,dr\,d\theta\,dz = \frac{19}{30}\pi.$$

c) (Harder...) Directly evaluate the surface integral

$$\int\int_{\partial W} \boldsymbol{V} \cdot \boldsymbol{n}\, dS$$

by splitting it into two pieces, one along the side of the closed cone and the other along the top. Confirm that this yields the same answer as that obtained in part (b).

Solution: The unit normal to the top surface is $\boldsymbol{n} = \boldsymbol{k}$ and $\boldsymbol{V}(x,y,1) = (x^3, y^3, 1)$, so

$$\int\int_{\partial W_{\text{top}}} \boldsymbol{V} \cdot \boldsymbol{n}\, dS = \text{Area}(\partial W_{\text{top}}) = \pi.$$

For the integral on the side of the cone, we parametrize ∂W_{side} by $\boldsymbol{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, r)$, $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Note that this parametrization gives us tangent vectors so that the normal *inwards*, not outwards, so we will have to flip the sign at the end of the calculation to get the correct contribution. Since

$$\boldsymbol{n}_{\text{in}}\, dS = d\boldsymbol{S} = \boldsymbol{T}_r \times \boldsymbol{T}_\theta\, dr\,d\theta = (r\cos\theta, -r\sin\theta, -r)\,dr\,d\theta,$$

we get that

$$\begin{aligned}\int\int_{\partial W_{\text{side}}} \boldsymbol{V} \cdot \boldsymbol{n}_{\text{in}}\, dS &= \int\int_{\partial W_{\text{side}}} (r^3\cos^3\theta, r^3\sin^3\theta, r) \cdot (r\cos\theta, -r\sin\theta, r)\, dr\,d\theta \\ &= \int_0^{2\pi} \int_0^1 \left(r^2 - r^4(\cos^4\theta + \sin^4\theta)\right) dr\,d\theta \\ &= \frac{11}{30}\pi.\end{aligned}$$

Here, we have used the identity $\int_0^{2\pi} (\cos^4\theta + \sin^4\theta)\,d\theta = 3\pi/2$. Putting this all together we again have that $\int\int_{\partial W} \boldsymbol{V} \cdot \boldsymbol{n}\, dS = \pi - 11\pi/30 = 19\pi/30$.

Question #1.4 (20 points)

One revolution of the helicoid is given by the parametrization

$$\mathbf{\Phi}(r,\theta) = (r\cos\theta, r\sin\theta, \theta), \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

a) Compute $\int\int_{\mathbf{\Phi}} 2r\,dS$.

Solution: Since $\boldsymbol{T}_r = \partial\mathbf{\Phi}/\partial r = (\cos\theta, \sin\theta, 0)$ and $\boldsymbol{T}_\theta = \partial\mathbf{\Phi}/\partial\theta = (-r\sin\theta, r\cos\theta, 1)$, we have that $\boldsymbol{T}_r \times \boldsymbol{T}_\theta = (\sin\theta, -\cos\theta, r)$. Therefore,

$$\int\int_{\mathbf{\Phi}} 2r\,dS = \int\int_{\mathbf{\Phi}} 2r\|\boldsymbol{T}_r \times \boldsymbol{T}_\theta\|dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{1+r^2}\,dr\,d\theta = 2\pi\Big(2\sqrt{2}-1\Big).$$

b) Compute $\int\int_{\mathbf{\Phi}} \boldsymbol{F}\cdot d\boldsymbol{S}$ where $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ with $(x,y,z) = \mathbf{\Phi}(r,\theta)$.

Solution:

$$\begin{aligned}\int\int_{\mathbf{\Phi}} \boldsymbol{F}\cdot d\boldsymbol{S} &= \int\int_{\mathbf{\Phi}} \boldsymbol{F}\cdot(\boldsymbol{T}_r \times \boldsymbol{T}_\theta)dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 (r\cos\theta, r\sin\theta, \theta)\cdot(\sin\theta, -\cos\theta, r)dr\,d\theta \\ &= \pi^2.\end{aligned}$$

Question #2.1 (30 points)

Consider the function

$$f(x,y) = -x^2 - y^2 + x + y + 4.$$

a) Find all critical points of f in $\mathbb{R}^2$, and classify them *using the second derivative test*.

Solution: [*This question appeared in almost identical form on Quiz 5.*] We first set the gradient of f equal to zero to find critical points:

$$0 = \nabla f(x,y) = (-2x+1, -2y+1).$$

This implies that the only critical point is $(1/2, 1/2)$. Since the Hessian is

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

the discriminant is $D(x,y) = \det H(x,y) = 4$. Then $D > 0$ and $f_{xx} < 0$ at the critical point implies that it is a local maximum.

b) Now suppose $S = \{(x,y)\colon x^2 + y^2 = 2\}$. Find extremizers of f on the domain S using any method of your choice (e.g., Lagrange multipliers, a parametrization of S, or a clever trick).

Solution: This can be done using Lagrange multipliers. Alternatively, one can use a parametrization of the curve $x^2 + y^2 = 2$ such as $x = \sqrt{2}\cos t$ and $y = \sqrt{2}\sin t$, $0 \le t < 2\pi$, but this yields a less straightforward approach for this problem. Define $g(x,y) = x^2 + y^2$. Then at constrained extrema of f we must have

$$\nabla f(x,y) = \lambda\nabla g(x,y), \qquad g(x,y) = 2$$

with multiplier λ. Since $\nabla g(x,y) = (2x, 2y)$, this yields the system of equations

$$\begin{aligned} 2(\lambda+1)x &= 1 \\ 2(\lambda+1)y &= 1 \\ x^2+y^2 &= 2. \end{aligned}$$

The first two equations imply that $x = y$, which upon substitution in the third equation gives that $y = \pm\, 1$. Substituting this backwards, we have constrained extrema at the points $(1,1)$ and $(-1,-1)$ at which f takes the value 4 and 0, respectively.

c) Use parts (a) and (b) to determine the location of the global maximum and global minimum of f on the domain $D = \{(x,y): x^2+y^2 \leq 2\}$.

Solution: We need only compare the values for f obtained at the critical points in the interior of the domain and on the boundary. Since $f(1/2,\ 1/2) = 9/2$ we have that the global maximum on D is achieved at $(1/2,\ 1/2)$ while the global minimum is achieved at $(-1,-1)$.

Question #2.2 (25 points)

Suppose the atmospheric pressure at a point (x,y,z), $z \geq 0$, is given by the scalar field

$$P(x,y,z) = \frac{1}{\pi}\sin(\pi y) - e^{-x}z.$$

a) At the point (0, 1, 2), what is the (unit) direction vector in which the pressure *decreases* the fastest?

Solution: Since $\nabla P(x,\ y,\ z) = \left(e^{-x}z,\ \cos(\pi y),\ -e^{-x}\right)$, the pressure at (0, 1, 2) decreases the fastest in the direction

$$\boldsymbol{n} = -\frac{\nabla P(0,1,2)}{\|\nabla P(0,1,2)\|} = \frac{1}{\sqrt{6}}(-2,1,1).$$

b) Isobaric surfaces—surfaces of constant pressure—are given by $P(x,y,z) = c$ for some constant c (the contours of these surfaces at sea level are what you typically see in weather forecasting maps). Find the equation for the tangent plane to the isobaric surface at the point $(0,1,2)$.

Solution: Since $\nabla P(0,1,2)$ is normal to the isobaric surface $P(x,\ y,\ z) = P(0,1,2) = -2$, the equation for the tangent plane is $\nabla P(0,1,2)\cdot((x,y,z)-(0,1,2)) = 0$, i.e.,

$$2x - y - z = -3.$$

c) (Harder...) The *baroclinity* of a fluid with density $\rho(x,y,z)$ is given by

$$\frac{1}{\rho^2}(\nabla\rho \times \nabla P)$$

and measures the misalignment between level surfaces of the density and pressure fields. Show that if λ is a thermodynamic function of ρ and P—that is, $\lambda(x,y,z) = f(\rho(x,y,z), P(x,y,z))$ for some f—then

$$\frac{1}{\rho^3}\nabla\lambda\cdot(\nabla\rho\times\nabla P)=0.$$

This is part of the proof of what is known as *Ertel's theorem*.

[Note: For those who are interested, baroclinity is *the major contributing factor* in the generation of weather systems affecting the polar and mid-latitude regions, including most of the United States. Large baroclinity in the polar regions leads to *baroclinic instability*, from which cyclones (low-pressure systems) and anti-cyclones (high-pressure systems) are 'shed' from the Arctic into the northern hemisphere. These low-pressure systems are associated to significant precipitation events that we observe in our day-to-day weather (e.g., showers, thunderstorms, blizzards).]

Solution: The difficulty of this problem only lies in noticing the importance of λ being a function of ρ and P, which are themselves functions of x, y, and z. By the chain rule,

$$\nabla\lambda=\frac{\partial f}{\partial\rho}\nabla\rho+\frac{\partial f}{\partial P}\nabla P$$

which is a vector that lies in the plane determined by the vectors $\nabla\rho$ and ∇P. Therefore, $\nabla\lambda\cdot(\nabla\rho\times\nabla P)=0$.

Question #2.3 (25 points)

The following two questions are unrelated.

a) Suppose $\boldsymbol{V} = y^2\boldsymbol{i} + 2xy\boldsymbol{j} + 2z\boldsymbol{k}$ and C is some oriented curve that connects the origin to the point $(1,1,1)$. Is this enough information to evaluate $\int_C \boldsymbol{V}\cdot d\boldsymbol{s}$, and if so, what is it? In addition, can you evaluate the path integral $\int_C \|\boldsymbol{V}\|ds$ without further information?

Solution: Since $\boldsymbol{V}=\nabla\phi$ with $\phi(x,y,z)=xy^2+z^2$, it is a gradient vector field and

$$\int_C \boldsymbol{V}\cdot d\boldsymbol{s}=\phi(1,1,1)-\phi(0,0,0)=2.$$

However, the path integral $\int_C \|\boldsymbol{V}\|ds$ cannot be evaluated without knowing the path C.

b) Calculate

$$\int\int_R (x+y)^2 e^{x-y}\,dx\,dy$$

where R is the region bounded by $x+y=1$, $x+y=4$, $x-y=-1$, and $x-y=1$. [Hint: Use the linear transformation $T(u,v)=(x,y)$ given by $x=(u+v)/2$ and $y=(u-v)/2$.]

Solution: Since the Jacobian of the transformation is $\partial(x,y)/\partial(u,v)=-1/2$,

$$\int\int_R (x+y)^2 e^{x-y}\,dx\,dy \;=\; \frac{1}{2}\int_{-1}^{1}\int_{1}^{4} u^2 e^v\,du\,dv=\frac{1}{3}\big(e-e^{-1}\big).$$

Question #2.4 (20 points)

We evaluate the distance between the point (1, 1, 1) and the plane $x - y + 2z = 5$ by the following two distinct methods:

a) Consider a vector joining $(1,1,1)$ to any chosen point on the plane. Compute the desired distance directly by projecting this vector in the *direction* of the normal vector to the plane.

Solution: A unit normal vector to the plane is $\boldsymbol{u} = \frac{1}{\sqrt{6}}(1,-1,2)$. Since the point $(4,1,1)$ lies in the plane, the desired distance is the length of the projection of $\boldsymbol{c} = (4,1,1) - (1,1,1) = (3,0,0)$ onto $\boldsymbol{u}$:

$$d = \|\mathrm{proj}_{\boldsymbol{u}}\, \boldsymbol{c}\| = |\boldsymbol{c} \cdot \boldsymbol{u}| = \sqrt{\frac{3}{2}}.$$

b) Use the method of Lagrange multipliers to extremize the squared distance function

$$d_{\text{squared}}(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$$

subject to an appropriate constraint (note that extremizing the *squared* distance between two arbitrary points is the same as extremizing the distance). Write down the set of equations to be solved, and explicitly find the solution. Use this to confirm your answer to part (a).

Solution: We seek to extremize d_{squared} with the constraint $x - y + 2z = 5$. Using Lagrange multipliers, the set of equations to be solved is

$$2(x-1, y-1, z-1) = \lambda(1,-1,2), \qquad x - y + 2z = 5.$$

Solving these equations, we find that $\lambda = 1$ so $(x,y,z) = (3/2, 1/2, 2)$ is the point at which d_{squared} is extremized. Therefore, the distance is

$$d = \sqrt{\left(\frac{3}{2} - 1\right)^2 + \left(\frac{1}{2} - 1\right)^2 + (2-1)^2} = \sqrt{\frac{3}{2}}.$$