M427L (55200), Homework #10

Due: 12:00pm, Wednesday, Nov. 02

Instructions: Questions are from the book "Vector Calculus, 5th ed." by Marsden and Tromba. Please show all your work, not only your final answer, to receive credit. Keep answers organized in the same order the problems have been assigned.

Path integrals (7.1)

p. 427-429, #2, 3, 5, 6, 13

Line integrals (7.2)

p. 447-451, #1, 2, 3, 4, 9, 11, 12, 14, 15, 16, 18

In addition:

- Consider the 2-D vector field

$$\boldsymbol{V}(x,y) = \frac{\Gamma}{2\pi r^2}(-y\boldsymbol{i} + x\boldsymbol{j}), \qquad (x,y) \neq (0,0)$$

  where $r^2 = x^2 + y^2$ and $\Gamma > 0$ is some constant. In fluid mechanics, $\boldsymbol{V}$ is the velocity field induced by a *point vortex* at the origin with strength Γ.

  a) Show that $\boldsymbol{V} = \nabla\phi$ with $\phi(x,y) = \frac{\Gamma}{2\pi}\arctan\left(\frac{y}{x}\right)$, for all $(x,y) \neq (0,0)$. Use this to demonstrate that the *circulation*

$$\int_C \boldsymbol{V} \cdot d\boldsymbol{s}$$

  is zero for any simple closed curve C **which does not encircle the origin**.

  b) Show that $\operatorname{curl}\boldsymbol{V} = \boldsymbol{0}$ for all $(x,y) \neq (0,0)$. Conclude that

$$\int\int_D (\operatorname{curl}\boldsymbol{V}) \cdot \boldsymbol{k}\, dA = 0$$

  for any region $D \subset \mathbb{R}^2$ **which does not contain the origin**. [Note: As we will discuss later, the fact that this integral is zero follows from the part (a) by Green's theorem.]

  c) Now let C_R be the circle of radius R centered at the origin, traversed in a counter-clockwise orientation. Compute $\int_{C_R} \boldsymbol{V} \cdot d\boldsymbol{s}$, and show that this value does not depend on R. Conclude that the circulation about the origin is nonzero by taking the limit $R \to 0$. [Note: This is what we mean by a *point* vortex: all of the circulation is concentrated at the origin!]

  d) Why does your answer to part (c) not contradict part (a)?