M427L (55200), Midterm #1 Solutions

Question #1 (25 points)

Suppose

$$\begin{aligned} \boldsymbol{u} &= (3, -1, 1) \\ \boldsymbol{v} &= (1, 2, 0) \\ \boldsymbol{w} &= (3, 1, 2). \end{aligned}$$

a) What is $\cos\theta$, where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$?

Solution:

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{1}{\sqrt{55}}.$$

b) Compute $|\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})|$. Geometrically, what does this positive number describe?

Solution: Since

$$\boldsymbol{v}\times\boldsymbol{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix} = (4, -2, -5),$$

$$|\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w})| = 9.$$

This is the volume of the parallelpiped with sides given by the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$.

c) Consider two parallel planes, both normal to $\boldsymbol{u}$, such that the point (1, 4, 6) lies in the first plane and $(5, 6, 7)$ in the second plane. What is the distance between these planes?

Solution: We need only find the distance between the point (5, 6, 7) and the first plane, since the planes are parallel. This can either be done directly using the distance formula or by using vector projection. The first plane has unit normal vector

$$\boldsymbol{n} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} = \frac{1}{\sqrt{11}}(3, -1, 1).$$

The vector that goes from the point $(1, 4, 6)$ in the plane to the external point $(5, 6, 7)$ is $\boldsymbol{a} = (4, 2, 1)$. Therefore, the distance between the planes is the magnitude of the vector projection $(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}$:

$$d = |\boldsymbol{a}\cdot\boldsymbol{n}| = \frac{11}{\sqrt{11}} = \sqrt{11}.$$

d) (Harder...) Suppose we are given two lines, the first having direction $\boldsymbol{v}$ and passing through $\boldsymbol{r}_0 = (0, 0, 0)$ and the second having direction $\boldsymbol{w}$ and passing through the point $\boldsymbol{s}_0 = (1, 3, -1)$. Their parametric equations are $\boldsymbol{r}(t) = \boldsymbol{r}_0 + t\boldsymbol{v}$ and $\boldsymbol{s}(t) = \boldsymbol{s}_0 + t\boldsymbol{w}$, $-\infty < t < \infty$, respectively. Compute the distance between these two lines.

Solution: First note that the points on the two lines which are closest to each other must lie on a line which is orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$ (the other way to see this is that there is a unique pair of parallel planes such that each contains one of the lines). We must therefore find the distance between the point $\boldsymbol{s}_0$ and the plane in the direction of $\boldsymbol{v}\times\boldsymbol{w}$ which contains the point $\boldsymbol{r}_0$. Letting $\boldsymbol{b}=\boldsymbol{s}_0-\boldsymbol{r}_0=(1,3,-1)$ and

$$\boldsymbol{m}=\frac{(\boldsymbol{v}\times\boldsymbol{w})}{\|\boldsymbol{v}\times\boldsymbol{w}\|}=\frac{1}{\sqrt{45}}(4,-2,-5)$$

we find that the desired distance is

$$d=|\boldsymbol{b}\cdot\boldsymbol{m}|=\frac{3}{\sqrt{45}}=\frac{1}{\sqrt{5}}.$$

Question #2 (10 points)

Spherical coordinates are given by

$$x=\rho\cos\theta\sin\phi,\qquad y=\rho\sin\theta\sin\phi,\qquad z=\rho\cos\phi.$$

If $u(x,y,z)$ is some known function given in Euclidean coordinates, compute $\partial u/\partial\phi$ in terms of $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$.

Solution: By the chain rule,

$$\frac{\partial u}{\partial\phi}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial\phi}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\phi}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial\phi}=\rho\cos\theta\cos\phi\,\frac{\partial u}{\partial x}+\rho\sin\theta\cos\phi\,\frac{\partial u}{\partial y}-\rho\sin\phi\,\frac{\partial u}{\partial z}.$$

Question #3 (25 points)

The height z of a mountain above the point (x,y) is given by

$$z=x(2-\sin y).$$

a) Starting at the point $(1,\ \pi/2)$, in what direction should one proceed to go *down* the fastest?

   Solution: The direction of fastest ascent at (x,y) is the gradient vector

   $$\nabla z(x,y)=(2-\sin y,-x\cos y).$$

   Therefore, the gradient at $(1,\pi/2)$ is $(1,0)$ and the direction of fastest descent is

   $$-\nabla z(1,\pi/2)=(-1,0).$$

b) At the point $(1,\pi/2)$, in what two directions can one go so that the elevation is increasing at a rate equal to 50% the rate of steepest ascent?

Solution: The rate of steepest ascent is $|\nabla z(1, \pi, 2)| = 1$, so we seek unit (direction) vectors $\boldsymbol{n} = (a,\ b)$ (with a, b to be determined) such that the directional derivative with respect to $\boldsymbol{n}$ is 1/2. That is,

$$a^2 + b^2 = 1 \qquad \text{and} \qquad 1/2 = \nabla z(1, \pi/2) \cdot \boldsymbol{n} = a.$$

This implies that $a = 1/2$ and $b = \pm\sqrt{3}/2$, so the desired directions are $\left(1/2,\ \sqrt{3}/2\right)$ and $\left(1/2, -\sqrt{3}/2\right)$.

c) Find the equation of the plane tangent to the mountain at the point $(1, \pi/2)$.

Solution: The tangent plane at a point $(x_0,\ y_0)$ on the graph $z = f(x,\ y)$ is given by the equation

$$z = f(x_0,\ y_0) + \nabla f(x_0,\ y_0) \cdot (x - x_0,\ y - y_0).$$

(This can be derived by noting that gradients are orthogonal to level surfaces, or directly by a first order Taylor expansion.) Therefore the tangent plane is given by the equation

$$z = 1 + (1, 0) \cdot (x - 1,\ y - \pi/2) = x,$$

that is, $x - z = 0$.

d) (Harder...) A particle follows a path on the mountain such that the corresponding path in the $(x,\ y)$ plane is given by $\boldsymbol{c}(t) = \left(t,\ t^2\right)$. How fast is the altitude changing at $t = \sqrt{\pi}$? [Hint: Consider the direction of the vector tangent to the path at $t = \sqrt{\pi}$.]

Solution: The path in the $(x,\ y)$ plane at the point $\boldsymbol{c}\left(\sqrt{\pi}\right) = \left(\sqrt{\pi}, \pi\right)$ has tangent vector $\boldsymbol{c}'\left(\sqrt{\pi}\right) = \left(1, 2\sqrt{\pi}\right)$. The rate of elevation change at $t = \sqrt{\pi}$ is therefore

$$\nabla z\left(\sqrt{\pi}, \pi\right) \cdot \frac{\boldsymbol{c}'\left(\sqrt{\pi}\right)}{\left\|\boldsymbol{c}'\left(\sqrt{\pi}\right)\right\|} = \left(2, \sqrt{\pi}\right) \cdot \frac{\left(1, 2\sqrt{\pi}\right)}{\sqrt{1 + 4\pi}} = \frac{2 + 2\pi}{\sqrt{1 + 4\pi}}.$$

Question #4 (25 points)

Consider the function

$$f(x, y) = x^2 y - \frac{1}{2}x^2 - \frac{1}{2}y^2$$

a) Find all critical points of f.

Solution: We seek (x, y) such that

$$(0, 0) = \nabla f(x, y) = \left(2xy - x, x^2 - y\right).$$

Therefore, $y = x^2$ and $(2y - 1)x = 0$. Plugging the first equation into the second yields that $\left(2x^2 - 1\right)x = 0$, so $x = 0$ (and $y = 0$) or $x = \pm 1/\sqrt{2}$ (and $y = 1/2$). So we have critical points at $(0, 0)$, $\left(1/\sqrt{2}, 1/2\right)$ and $\left(-1/\sqrt{2}, 1/2\right)$.

b) Classify each of these critical points as local maxima, minima, or saddle points using the second derivative test.

Solution: The Hessian matrix is

$$H(x,y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 2y-1 & 2x \\ 2x & -1 \end{bmatrix}$$

which gives the discriminant

$$D(x,y) = \det\left(H(x,y)\right) = 1 - 2y - 4x^2.$$

Since $D(0,0) = 1 > 0$ and $f_{xx}(0,0) = -1$, the point $(0,0)$ is a local maximum. For the two remaining critical points, $D\left(\pm 1/\sqrt{2}, 1/2\right) = -2 < 0$ implies that these are saddle points.

c) (Harder...) Find the global maximum and minimum of f restricted to the domain

$$D = \left\{(x,y)\colon x^2 + y^2 \leq 6\right\}.$$

[Hint: To look for extrema on the boundary ∂D of D, use either a parametrization of the boundary or the method of Lagrange multipliers. Note that $f(x,\,y) = x^2 y - 3$ on ∂D (since $x^2 + y^2 = 6$ on ∂D) which simplifies the expression you need to extremize!]

Solution: We can do this using a parametrization of the boundary as follows. Let $x = \sqrt{3}\cos t$ and $y = \sqrt{3}\sin t$ for $0 \leq t < 2\pi$. Then on ∂D, the function to be extremized is

$$h(t) = f(x(t), y(t)) = 3\sqrt{3}\cos^2 t \sin t - 3.$$

We seek t such that

$$0 = h'(t) = 3\sqrt{3}\left(-2\cos t \sin^2 t + \cos^3 t\right) = 3\sqrt{3}\cos t\left(-2\sin^2 t + \cos^2 t\right).$$

Therefore, either $\cos t = 0$ or $|\cos t| = \sqrt{2}\,|\sin t|$. The first expression implies that $t = \pi/2$ or $3\pi/2$, and the second that $t = \pi/6$, $5\pi/6$, $7\pi/6$ or $11\pi/6$. These correspond to the six candidate points $\left(0, \pm\sqrt{6}\right)$, $\left(\pm 2, \pm\sqrt{2}\right)$. Since

$$f\left(0, \pm\sqrt{6}\right) = -3, \qquad f\left(\pm 2, \sqrt{2}\right) = 4\sqrt{2} - 3, \qquad f\left(\pm 2, -\sqrt{2}\right) = -4\sqrt{2} - 3$$

and maximum value of f in the interior of D is $f(0,0) = 0$, we conclude that the global maximum value is $4\sqrt{2} - 3$ and the global minimum value is $-4\sqrt{2} - 3$ (both of which are achieved on ∂D).

Alternatively, we can extremize $f(x,y) = x^2 y - 3$ with the constraint $g(x,y) = x^2 + y^2 = 6$ by using Lagrange multipliers. We seek x, y, and λ such that $\nabla f = \lambda \nabla g$, giving the set of equations

$$\begin{aligned} 2xy &= 2\lambda x \\ x^2 &= 2\lambda y \\ x^2 + y^2 &= 6. \end{aligned}$$

First note that if $x = 0$ then $\lambda = 0$ and $y = \pm\sqrt{6}$, giving two candidate points $\left(0, \pm\sqrt{6}\right)$. Alternatively, if $x \neq 0$ then $\lambda \neq 0$ and $y \neq 0$. In this case we can divide the first equation by the second and rearrange terms to get that $x^2 = 2y^2$. Inserting this into the third equation implies that $x = \pm 2$ and $y = \pm\sqrt{2}$, yielding the four candidate points $\left(\pm 2, \pm\sqrt{2}\right)$. This yields the global minimum and maximum as before.

Question #5 (15 points)

Determine if the following statements are true or false. Justify your answers.

a)

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^4y^2 + x^2y^4} = 1.$$

Solution: This is false. An easy way to see this is that along paths $x = y$, the limiting value is $1/2$. Furthermore, it can be seen that the limiting value is not 1 even when traveling along the x or y axes as follows. Note that the function can be rewritten as $xy/\left(x^2 + y^2\right)$ after cancelling the term x^2y^2 from both the numerator and denominator. Along the path $x = 0$ or $y = 0$,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = 0$$

so one has different limiting values along different paths and the limit does not exist at $(0, 0)$.

b) The length l, width w, and height h of a rectangular box *with no top* which has fixed surface area 16 and maximal volume must satisfy the set of equations

$$\frac{wh}{w + 2h} = \frac{lh}{l + 2h} = \frac{lw}{2l + 2w} \qquad \text{and} \qquad lw + 2lh + 2wh = 16.$$

Solution: This is true. We seek to maximize the function

$$V(l, w, h) = lwh$$

subject to the constraint $S(l, w, h) = lw + 2lh + 2wh = 16$. By the method of Lagrange multipliers we therefore seek (l, w, h) and λ such that $\nabla V = \lambda \nabla S$, which yields the equations

$$\begin{aligned} wh &= \lambda(w + 2h) \\ lh &= \lambda(l + 2h) \\ lw &= \lambda(2l + 2w) \\ lw + 2lh + 2wh &= 16. \end{aligned}$$

This is exactly the set of equations given above after eliminating λ.