M427L (55200), Midterm #2 Solutions

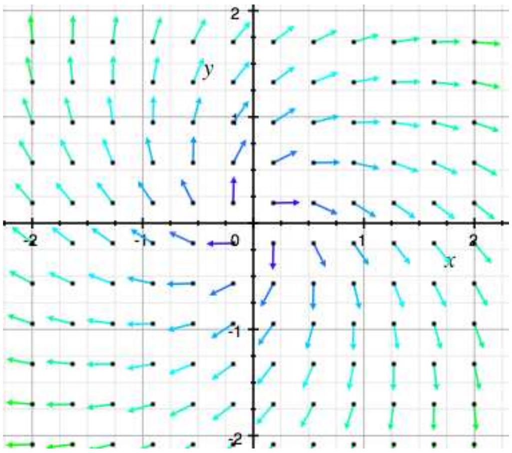
**Question #1 (25 points)**

Consider the 2-D vector field

$$\boldsymbol{V}(x,y) = (x+y)\boldsymbol{i} + (y-x)\boldsymbol{j}.$$

a) Sketch the vector field, only drawing enough scaled vectors necessary to give basic picture. Is $\boldsymbol{c}(t) = \left(e^t \cos t, e^t \sin t\right)$ a flow path for $\boldsymbol{V}$?

**Solution:**




No, $\boldsymbol{c}(t)$ is not a flow path since $\boldsymbol{c}'(t) = \left(-e^t \sin t + e^t \cos t, e^t \sin t + e^t \cos t\right)$ but $\boldsymbol{V}(\boldsymbol{c}(t)) = \left(e^t \sin t + e^t \cos t, e^t \sin t - e^t \cos t\right)$ so $\boldsymbol{c}'(t) \neq \boldsymbol{V}(\boldsymbol{c}(t))$. This can also be seen by noting that $\boldsymbol{c}(t)$ is spirals out from the origin in a counterclockwise fashion and therefore $\boldsymbol{c}'(t)$ is not parallel to $\boldsymbol{V}$.

b) Compute $\operatorname{div} \boldsymbol{V}$ and $\operatorname{curl} \boldsymbol{V}$.

**Solution:** $\operatorname{div} \boldsymbol{V} = 2$ and $\operatorname{curl} \boldsymbol{V} = -2$ for all $(x,y) \in \mathbb{R}^2$.

c) Suppose $\boldsymbol{V}$ is the velocity field of a fluid. At time $t = 0$, a small droplet of red dye is injected into the fluid at the point $(0,0)$ and a small droplet of blue dye of the same size is placed at $(1, 1)$. A short time later, what is the ratio of the sizes of the blue and red droplets? Has the blue droplet rotated relative to its center, and if so, in which direction (clockwise or counterclockwise)? Justify your answers.

**Solution:** Since the divergence is constant, the local rate of expansion/contraction is the same everywhere in the plane. Therefore, the ratio of the sizes of the blue and red droplets a short time later is 1 since they will be the same size. The blue droplet will have rotated in a clockwise direction about its center since the curl is negative at $(1,1)$.

d) Does there exist a vector field $\boldsymbol{W}$ such that $\nabla \times \boldsymbol{W} = \boldsymbol{V}$? Why or why not?

**Solution:** No, since if such a $\boldsymbol{W}$ exists then we must have that $\nabla \cdot (\nabla \times \boldsymbol{W}) = \operatorname{div} \boldsymbol{V} = 0$, which is clearly not true from part (b).

e) Evaluate each of the following quantities if it makes sense, and if it doesn't make sense indicate why not.

i. $\operatorname{curl}\operatorname{div}\boldsymbol{V}$

ii. $\nabla\times\operatorname{grad}\big(\|\boldsymbol{V}\|^2\big)$

iii. [Harder...] $(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}$

**Solution:** (i) does not make sense since the curl acts on vector fields, but $\operatorname{div}\boldsymbol{V}$ is a scalar field; (ii) 0, since curls of gradients of scalar fields are zero; (iii) this makes sense by noting that $\boldsymbol{V}\cdot\nabla$ is a differential operator that acts on vector fields to give

$$(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}=\left((x+y)\frac{\partial}{\partial x}+(y-x)\frac{\partial}{\partial y}\right)((x+y)\boldsymbol{i}+(y-x)\boldsymbol{j})=2y\boldsymbol{i}-2x\boldsymbol{j}.$$

**Question #2 (20 points)**

Let $P$ be the parallelogram in the $xy$-plane with vertices $(0,0)$, $(1,2)$, $(3,4)$, and $(2,2)$. Find the integral

$$\int\int_P xy\,dx\,dy$$

using the following steps.

a) Consider the linear transformation $T(u,v)=(x,y)$ given by $x=u-v,\ y=2u-v$. What is the Jacobian of this transformation?

**Solution:**

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1 & 2\\ -1 & -1\end{vmatrix}=1.$$

b) Rewrite the above integral in terms of $u$ and $v$, with appropriate boundaries of integration. Evaluate this to get the desired answer.

**Solution:** The inverse of the transformation $T^{-1}\colon(x,y)\mapsto(u,v)$ takes the set $P$ in the $xy$-plane to the rectangle $Q=[0,1]\times[-2,0]$ in the $uv$-plane. Then,

$$\int\int_P xy\,dx\,dy=\int\int_Q(u-v)(2u-v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv=\int_{-2}^{0}\int_0^1\big(2u^2-3uv+v^2\big)du\,dv=7.$$

**Question #3 (20 points)**

Let $C$ be the three-dimensional curve traced out by the path $\boldsymbol{c}(t)=(t,t\sin t,t\cos t)$, $0\le t\le 4\pi$.

a) With $\boldsymbol{F}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$, find

$$\int_C\boldsymbol{F}\cdot d\boldsymbol{s},$$

where $C$ is traversed in the orientation **opposite** to $\boldsymbol{c}(t)$.

**Solution:** Since $\boldsymbol{c}'(t) = (1, \sin t + t\cos t, \cos t - t\sin t)$,

$$\int_{\boldsymbol{c}} \boldsymbol{F}\cdot d\boldsymbol{s} = \int_0^{4\pi} \boldsymbol{F}(\boldsymbol{c}(t))\cdot \boldsymbol{c}'(t)\,dt = \int_0^{4\pi} \left[t + t\left(\sin^2 t + \cos^2 t\right)\right] dt = 16\pi^2.$$

Therefore, since the orientation is reversed we have that $\int_C \boldsymbol{F}\cdot d\boldsymbol{s} = -16\pi^2$.

b) Write an expression for the arc length of $C$. Simplify this expression as much as possible, but do not evaluate it.

**Solution:** The arc length is

$$\int_C ds = \int_0^{4\pi} \sqrt{1^2 + (\sin t + t\cos t)^2 + (\cos t - t\sin t)^2}dt = \int_0^{4\pi} \sqrt{2+t^2}\,dt.$$

c) Compute $\int_C x\,ds$.

**Solution:**

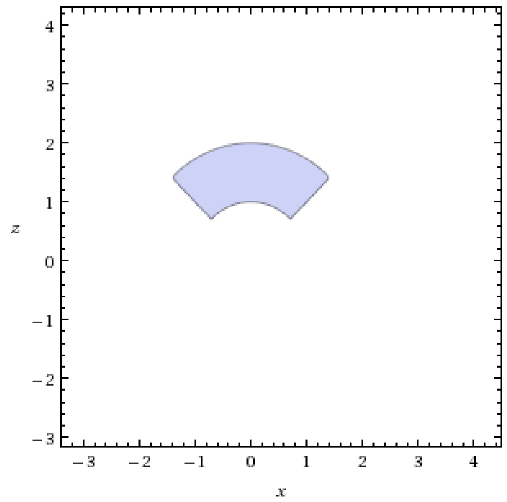
$$\int_C x\,ds = \int_0^{4\pi} t\sqrt{2+t^2}\,dt = \frac{1}{2}\int_0^{2+16\pi^2} \sqrt{u}\,du = \frac{\left(2+16\pi^2\right)^{3/2}}{3}.$$

**Question #4 (20 points)**

Let $D$ be the region defined by $1 \le x^2 + y^2 + z^2 \le 4$ and $z \ge \sqrt{x^2+y^2}$. Note that $D$ is rotationally symmetric about the $z$-axis.

a) Sketch the region in the $xz$-plane that intersects $D$.

**Solution:**




b) Compute the integral

$$\int\int\int_D \left(x^2+y^2+z^2\right)^{1/2} \exp\left[\left(x^2+y^2+z^2\right)^2\right] dx\,dy\,dz$$

by first writing it in terms of spherical coordinates with the appropriate boundaries of integration. [Hint: Remember that $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\rho^2 \sin\phi$.]

**Solution:** We get that the integral is

$$\int_0^{\pi/4} \int_0^{2\pi} \int_1^2 \rho^3 e^{\rho^4} \sin\phi \, d\rho \, d\theta \, d\phi = \frac{\pi}{2}\left(1 - \frac{\sqrt{2}}{2}\right)\left(e^{16} - e\right).$$

**Question #5 (15 points)**

a) If $\boldsymbol{c}(t) = \left(t^3/3, \cos(\pi t), t\right)$, $0 \leq t \leq 1$, find

$$\int_{\boldsymbol{c}} xy\,dz + xz\,dy + yz\,dx.$$

**Solution:** We can write this as $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$ for a gradient vector field $\boldsymbol{F} = \nabla f$ with $f(x, y, z) = xyz$. Therefore, the line integral only depends on the endpoints and we easily get the answer as $f(1/3, -1, 1) - f(0, 1, 0) = -1/3$.

b) [Harder...] Consider the vector fields

$$\boldsymbol{V}_{\text{vortex}}(x, y) = \frac{1}{r^2}(-y\boldsymbol{i} + x\boldsymbol{j}) = \nabla\left(\arctan\left(\frac{y}{x}\right)\right), \qquad (x, y) \neq (0, 0)$$

and

$$\boldsymbol{V}_{\text{radial}}(x, y) = \frac{1}{r^2}(x\boldsymbol{i} + y\boldsymbol{j}) = \nabla\left(\ln\left(\sqrt{x^2 + y^2}\right)\right), \qquad (x, y) \neq (0, 0),$$

where $r^2 = x^2 + y^2$. Although these fields appear similar superficially and both have a singularity at the origin, they are actually quite different. Show that $\boldsymbol{V}_{\text{vortex}}$ has zero circulation for all paths *except* those about the origin, while $\boldsymbol{V}_{\text{radial}}$ has zero circulation for all paths *including* those about the origin.

**Solution:** We know that if a vector field is the gradient of a $C^1$ scalar function in a domain $D$, the line integral over any closed path in $D$ is necessarily zero. Therefore, the only thing that remains to be shown is that line integrals of $\boldsymbol{V}_{\text{vortex}}$ encircling the origin are nonzero, while those for $\boldsymbol{V}_{\text{radial}}$ vanish. Note that it is only necessary to check this along circular paths about the origin.

Let $C_R$ be the circular path of radius $R$ centered at the origin, traversed counterclockwise. Using the parametrization $\boldsymbol{c}(t) = (R\cos t, R\sin t)$ with $0 \leq t \leq 2\pi$, we have that

$$\int_{C_R} \boldsymbol{V}_{\text{vortex}} \cdot d\boldsymbol{s} = \int_0^{2\pi} \frac{1}{R^2}(-R\sin t, R\cos t) \cdot (-R\sin t, R\cos t)\,dt = 2\pi$$

and

$$\int_{C_R} \boldsymbol{V}_{\text{radial}} \cdot d\boldsymbol{s} = \int_0^{2\pi} \frac{1}{R^2}(R\cos t, R\sin t) \cdot (-R\sin t, R\cos t)\,dt = 0.$$

Therefore, the circulation about the origin is nonzero for $\boldsymbol{V}_{\text{vortex}}$ but vanishes for $\boldsymbol{V}_{\text{radial}}$.