M427L (55200), Quiz #1 Solutions

Question #1 (4 pts.)

Does the line $\boldsymbol{v} = (2, -2, 1) + t(1, 1, 1)$, $t \in \mathbb{R}$, intersect the plane given by $2x - 4y - z = 3$? If so, at what point?

Solution: [*Question based on p. 22, #19, 22.*] The line satisfies the equations

$$\begin{aligned} x &= 2 + t \\ y &= -2 + t \\ z &= 1 + t \end{aligned}\ , \qquad t \in \mathbb{R}.$$

Subsituting these into the equation for the plane in order to find an intersection, we have that

$$(4 + 2t) + (8 - 4t) + (-1 - t) = 3,$$

so $t = 8/3$. Therefore, the line intersects the plane at $(2, -2, 1) + \frac{8}{3}(1, 1, 1) = \frac{1}{3}(14, 2, 11)$.

Question #2 (4 pts.)

Suppose $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{k}$.

a) Find $\cos\theta$, where θ is the angle between the two vectors.

   Solution: [*Question based on p. 36, #3.*] We have that $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\theta$. Note that $\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 4 = -3$, $\|\boldsymbol{a}\| = 3\sqrt{2}$, and $\|\boldsymbol{b}\| = \sqrt{2}$. Therefore,

$$\cos\theta = -\frac{1}{2}.$$

b) Express $\boldsymbol{b}$ as a sum of two vectors, one of which is parallel to $\boldsymbol{a}$ and the other which is orthogonal to $\boldsymbol{a}$. (Hint: Use vector projection. If you've forgotten the formula, first rederive it using a diagram.)

   Solution: [*Question based on p. 36, #14.*] Draw a picture with the tails of the two vectors at the same point. Denoting $\boldsymbol{c} = \mathbf{proj}_{\boldsymbol{a}}\boldsymbol{b}$ as the vector projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$, we know

$$\boldsymbol{c} = \left(\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|^2}\right)\boldsymbol{a} = -\frac{1}{6}(\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k})$$

   is parallel to $\boldsymbol{a}$. Furthermore, $\boldsymbol{b} - \boldsymbol{c} = \frac{1}{6}(7\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k})$ is orthogonal to $\boldsymbol{a}$. Therefore the desired decomposition is

$$\boldsymbol{b} = -\frac{1}{6}(\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}) + \frac{1}{6}(7\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}).$$