M427L (55200), Quiz #11 Solutions

Question #1 (4 pts.)

Consider the parametrized surface S given by

$$\boldsymbol{\Phi}(u,v) = u\cos v\,\boldsymbol{i} + u\sin v\,\boldsymbol{j} + u\boldsymbol{k}, \qquad -\infty < u < \infty, \quad 0 \le v < 2\pi.$$

a) Where is the surface S regular? That is, for which (x, y, z) is $\boldsymbol{T}_u \times \boldsymbol{T}_v \neq \boldsymbol{0}$?

Solution: [*Question based on p. 460, #17.*] Since

$$\boldsymbol{T}_u = \cos v\,\boldsymbol{i} + \sin v\,\boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{T}_v = -u\sin v\,\boldsymbol{i} + u\cos v\,\boldsymbol{j},$$

we have that

$$\boldsymbol{T}_u \times \boldsymbol{T}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = -u\cos v\,\boldsymbol{i} + u\sin v\,\boldsymbol{j} + u\boldsymbol{k} \neq \boldsymbol{0} \qquad \text{if } u \neq 0.$$

Therefore, the surface is regular everywhere except at the point $\boldsymbol{\Phi}(0, v) = (0, 0, 0)$.

b) Find the equation for the tangent plane to S at $(3, 0, 3)$.

Solution: [*Question based on p. 459, #2, 7.*] The point $\boldsymbol{r}_0 = (3, 0, 3)$ corresponds to $(u, v) = (3, 0)$. A normal vector to the tangent plane to S at $(3, 0, 3)$ is then given by

$$\boldsymbol{n} = (\boldsymbol{T}_u \times \boldsymbol{T}_v)(3, 0) = -3\boldsymbol{i} + 3\boldsymbol{k}$$

so the vector equation for the plane is $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$. In coordinate form this is the plane $x = z$.

c) Give an equation in terms of x, y, and z for S. What is the name for this type of surface? Use this to confirm your answers to parts (a) and (b).

Solution: [*Question based on p. 459, #11, 12.*] Since $x = u\cos v$, $y = u\sin v$, and $z = u$, we can write S as

$$x^2 + y^2 = z^2.$$

This is the equation for a double cone. In particular, since the cone has a sharp point at the origin it is not regular there, as was shown in part (a). Additionally, it is easy to see that the plane $x = z$ is tangent to any point on the cone that lies in the xz-plane, which verifies the answer to part (b).

Question #2 (4 pts.)

The unit sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ can be described parametrically by

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$$

with $0 \leq \phi \leq \pi$ and $0 \leq \theta < 2\pi$. One can verify that

$$\frac{\partial(x,y)}{\partial(\phi,\theta)} = \sin\phi\cos\phi, \quad \frac{\partial(y,z)}{\partial(\phi,\theta)} = \sin^2\phi\cos\theta, \quad \frac{\partial(x,z)}{\partial(\phi,\theta)} = \sin^2\phi\sin\theta.$$

Use this to compute the surface integral

$$\int\int_S z^4 \, dS.$$

Solution: [*Question based on p. 472, #14, p. 480, #6, and example done in lecture.*] In parametric form, the surface area element is

$$dS = \|\boldsymbol{T}_\phi \times \boldsymbol{T}_\theta\| d\phi d\theta = \sqrt{\left(\frac{\partial(x,y)}{\partial(\phi,\theta)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(\phi,\theta)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(\phi,\theta)}\right)^2} \, d\phi d\theta.$$

Since

$$\begin{aligned} \left(\frac{\partial(x,y)}{\partial(\phi,\theta)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(\phi,\theta)}\right)^2 + \left(\frac{\partial(x,z)}{\partial(\phi,\theta)}\right)^2 &= \sin^2\phi\cos^2\phi + \sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta \\ &= \sin^2\phi\big(\cos^2\phi + \sin^2\phi\big(\cos^2\theta + \sin^2\theta\big)\big) \\ &= \sin^2\phi, \end{aligned}$$

the area element simplifies to $dS = \sin\phi \, d\phi d\theta$. Therefore, we find that

$$\int\int_S z^4 dS = \int_0^{2\pi} \int_0^{\pi} \cos^4\phi\sin\phi \, d\phi d\theta = 2\pi\left[-\frac{1}{5}\cos^5\phi\right]_0^{\pi} = \frac{4}{5}\pi.$$