M427L (55200), Quiz #5 Solutions

**Question #1 (2 pts.)**

Design a cylindrical soup can which has volume $c$ (where $c$ is some constant) and uses a minimal amount of metal. That is, simply *write a system of equations* for the radius $r$ and height $h$ of the can which will yield a solution to this minimization problem. ***Show your work, and do not solve for r and h!***

**Solution:** [*Question based on p. 244, #13.*] We seek to minimize the surface area $S(r,\, h) = 2\pi r h + 2\pi r^2$ of the can subject to the constraint $V(r, h) = c$, where $V(r, h) = \pi r^2 h$ is the volume of the can. Then

$$\nabla S(r, h) = (2\pi h + 4\pi r, 2\pi r), \qquad \nabla V(r, h) = \left(2\pi r h, \pi r^2\right)$$

and the method of Lagrange multipliers implies that we must solve the system of equations

$$\begin{aligned} 2\pi h + 4\pi r &= 2\lambda\pi r h \\ 2\pi r &= \lambda\pi r^2 \\ \pi r^2 h &= c \end{aligned}$$

to find the optimal design.

**Question #2 (6 pts.)**

Consider the function

$$f(x, y) = -x^2 - y^2 + \sqrt{3}\, x + y + 4.$$

a) (*2 pts.*) Find all critical points of $f$ in $\mathbb{R}^2$, and classify them using the second derivative test.

**Solution:** [*Question based on problem from lecture.*] We first set the gradient of $f$ equal to zero to find critical points:

$$0 = \nabla f(x, y) = \Big(-2x + \sqrt{3}, -2y + 1\Big).$$

This implies that the only critical point is $\left(\sqrt{3}/2, 1/2\right)$. Since the Hessian is

$$H(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix},$$

the discriminant is $D(x, y) = \det H(x, y) = 4$. Then $D > 0$ and $f_{xx} < 0$ at the critical point implies that it is a local maximum.

b) (*3 pts.*) Now suppose $S = \left\{(x, y) \colon x^2 + y^2 = 4\right\}$. Find extremizers of $f$ on the domain $S$.

**Solution:** [*Question based on problem from lecture*.] This can be done using Lagrange multipliers. Alternatively, one can use a parametrization of the curve $x^2 + y^2 = 4$ such as $x = 2\cos t$ and $y = 2\sin t$, $0 \le t < 2\pi$, but this yields a less straightforward approach for this problem. Define $g(x, y) = x^2 + y^2$. Then at constrained extrema of $f$ we must have

$$\nabla f(x, y) = \lambda \nabla g(x, y), \qquad g(x, y) = 4$$

with multiplier $\lambda$. Since $\nabla g(x, y) = (2x, 2y)$, this yields the system of equations

$$\begin{aligned} 2(\lambda + 1)x &= \sqrt{3} \\ 2(\lambda + 1)y &= 1 \\ x^2 + y^2 &= 4. \end{aligned}$$

The first two equations imply that $x = \sqrt{3}y$, which upon substitution in the third equation gives that $y = \pm 1$. Substituting this backwards, we have constrained extrema at the points $\left(\sqrt{3}, 1\right)$ and $\left(-\sqrt{3}, -1\right)$ at which $f$ takes the value 4 and $-4$, respectively.

c) (*1 pt.*) Use parts (a) and (b) to determine the location of the global maximum and global minimum of $f$ on the domain $D = \left\{(x, y) \colon x^2 + y^2 \le 4\right\}$.

**Solution:** We need only compare the values for $f$ obtained at the critical points in the interior of the domain and on the boundary. Since $f\left(\sqrt{3}/2,\, 1/2\right) = 5$ we have that the global maximum on $D$ is achieved at $\left(\sqrt{3}/2,\, 1/2\right)$ while the global minimum is achieved at $\left(-\sqrt{3}, -1\right)$.