M427L (55200), Quiz #6 Solutions

Question #1 (4 pts.)

Let $\boldsymbol{c}(t)$ be position of an object in $\mathbb{R}^3$ at time t. Assume that $\boldsymbol{c}$ is twice differentiable. Recall that the tangent vector $\boldsymbol{v}(t)=\boldsymbol{c}'(t)$ is its *velocity*, $\boldsymbol{a}(t)=\boldsymbol{v}'(t)$ is its *acceleration*, and $\|\boldsymbol{v}(t)\|$ is its *speed*.

a) (*2 pts.*) If the velocity of an object is always perpendicular to its position vector, what type of trajectory does the object necessarily trace out? Justify your answer.

Solution: [*Question based on p. 274, #14 and problem done in lecture.*] If $\boldsymbol{v}(t)=\boldsymbol{c}'(t)$ is always perpendicular to $\boldsymbol{c}(t)$, we must have that $\boldsymbol{c}(t)\cdot\boldsymbol{c}'(t)=0$ for all $t\in\mathbb{R}$. By the product rule this implies that

$$\frac{d}{dt}\left(\|\boldsymbol{c}(t)\|^2\right)=2\boldsymbol{c}(t)\cdot\boldsymbol{c}'(t)=0,$$

i.e., the position vector has constant magnitude. Therefore, the path must necessarily be a circle about the origin.

b) (*2 pts.*) If the acceleration of an object is always perpendicular to its velocity, can its speed ever change? Demonstrate why or why not.

Solution: [*Question based on p.274, #13.*] This is very similar to part (a). If $\boldsymbol{a}(t)=\boldsymbol{v}'(t)$ is always perpendicular to $\boldsymbol{v}(t)$, we must have that $\boldsymbol{v}(t)\cdot\boldsymbol{v}'(t)=0$ for all t, which implies

$$\frac{d}{dt}\left(\|\boldsymbol{v}(t)\|^2\right)=2\boldsymbol{v}(t)\cdot\boldsymbol{v}'(t)=0.$$

Therefore, the speed is constant and cannot change.

Question #2 (4 pts.)

Consider the path

$$\boldsymbol{c}(t)=-3\sin\frac{t}{2}\,\boldsymbol{i}+t^{3/2}\boldsymbol{j}+3\cos\frac{t}{2}\,\boldsymbol{k},\qquad t\geq 0.$$

Find the arc length of the curve between the points $\left(-3,\pi^{3/2},0\right)$ and $(0,0,3)$.

Solution: [*Question based on p. 281, #3.*] Note that

$$\boldsymbol{c}'(t)=-\frac{3}{2}\left(\cos\frac{t}{2}\,\boldsymbol{i}-t^{1/2}\boldsymbol{j}+\sin\frac{t}{2}\,\boldsymbol{k}\right)$$

and

$$\|\boldsymbol{c}'(t)\|=\frac{3}{2}\sqrt{\cos^2\left(\frac{t}{2}\right)+t+\sin^2\left(\frac{t}{2}\right)}=\frac{3}{2}\sqrt{1+t}.$$

The path passes through the points $\left(-3,\,\pi^{3/2},\,0\right)$ and $(0,\,0,\,3)$ at $t=\pi$ and $t=0$, respectively. Therefore, the arc length between these two points is

$$L(\boldsymbol{c})=\int_0^{\pi}\|\boldsymbol{c}'(t)\|dt=\frac{3}{2}\int_0^{\pi}\sqrt{1+t}\,dt=\frac{3}{2}\int_1^{1+\pi}\sqrt{u}\,du=(1+\pi)^{3/2}-1.$$