M427L (55200), Quiz #7 Solutions

Question #1 (6 pts.)

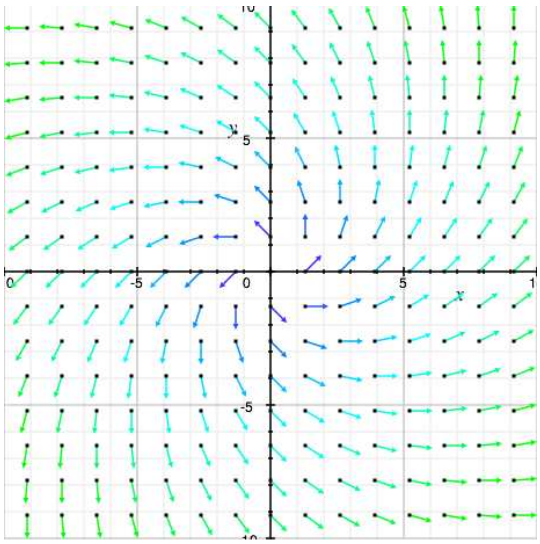
Suppose

$$\boldsymbol{V}(x, y, z) = (x-y)\boldsymbol{i} + (x+y)\boldsymbol{j}.$$

[Hint: It may be useful to think of $\boldsymbol{V} = \boldsymbol{W}_1 + \boldsymbol{W}_2$, where the vector fields $\boldsymbol{W}_1 = x\boldsymbol{i} + y\boldsymbol{j}$ and $\boldsymbol{W}_2 = -y\boldsymbol{i} + x\boldsymbol{j}$ were discussed at length in lecture.]

a) Sketch the vector field $\boldsymbol{V}$ in 2-D for $z = 0$ (only draw enough scaled vectors necessary to give basic picture).

Solution: [*Question based on p. 311-313, #2, 6, 31 and problems done in lecture.*] The vector field is depicted below, with the size of vectors indicated by color (blue is smaller, green is larger).



b) Compute $\operatorname{div}\boldsymbol{V}$.

Solution:

$$\operatorname{div}\boldsymbol{V} = \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(x+y) = 2 \qquad \text{for all } (x, y, z).$$

c) Compute $\operatorname{curl}\boldsymbol{V}$.

Solution:

$$\operatorname{curl}\boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x-y & x+y & 0 \end{vmatrix} = 2\boldsymbol{k} \qquad \text{for all } (x, y, z).$$

d) If a very small box of points is placed in the vector field at the point (1, 1, 1), will the volume of the box a short time later be larger, smaller, or the same? Will the orientation of the box about its center remain the same or will it have rotated? Justify your answers.

Solution: The volume of the box will be larger since the divergence of $\boldsymbol{V}$ at (1, 1, 1) is positive. Its orientation will be rotated with respect to the x-y plane since the curl at (1, 1, 1) is nonzero and points in the z-direction. In fact, both of these statements hold true everywhere and not only at the particular point $(1, 1, 1)$.

e) Is $\boldsymbol{V}$ a gradient vector field? If so, find its scalar potential and if not, justify your answer.

Solution: No, it is not a gradient vector field. This is easily seen by noting that its curl is nonzero. Explicitly, suppose $\boldsymbol{V} = (x - y, x + y) = \nabla f$ for some C^2 scalar function f. Then $\partial f/\partial x = x - y$ and $\partial f/\partial y = x + y$, so equality of mixed partials would tell us that

$$-1 = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 1.$$

Since this is obviously not true, $\boldsymbol{V}$ cannot be a gradient vector field.

Question #2 (2 pts.)

If $\boldsymbol{W}$ is a vector field and ϕ is a scalar field, indicate whether each of the five following expressions is a vector field, a scalar field, or doesn't make sense. Please note that you *do not* need to simplify or compute any of them!

i. $\nabla \cdot (\nabla \times \boldsymbol{W})$

ii. $\nabla \times (\nabla \cdot \boldsymbol{W})$

iii. $\operatorname{div} \operatorname{div} \boldsymbol{W}$

iv. $\nabla \times (\nabla \times \nabla \phi)$

v. $\operatorname{div} \operatorname{grad} \phi - \operatorname{grad} \operatorname{div} \boldsymbol{W}$

Solution: (i) scalar field, (ii) doesn't make sense since cannot take curl of a scalar field, (iii) doesn't make sense since cannot take divergence of a scalar field, (iv) vector field, (v) doesn't make sense since first term is scalar-valued while second term is vector-valued.