B: Sample Final Exam Questions With Solutions

Problem 1

(a) *The concentration of a toxic chemical in the xy-plane at position (x, y) is given by $c(x, y) = e^{-x}(xy+3-\sin(x-y))$. If an environmental worker is at $(x, y) = (1, 1)$, in what direction must she go to decrease the concentration as fast as possible?*

(b) *The environmental worker decides to follow the flow line of the vector field defined by $\mathbf{F}(x, y) = 3x\mathbf{i} + 2\mathbf{j}$, again starting at $(1, 1)$. How fast is the concentration of the chemical changing as the worker starts along the flow line, as a function of time t along the flow line ($t = 0$ corresponds to the starting position)?*

(c) *What is the acceleration of the worker following the path in (b) at $t = 0$?*

Solution.

(a) To decrease c as quickly as possible, one should go in the direction $-\nabla c(1, 1)$. Now,

$$\nabla c = \left(-e^{-x}(xy + 3 - \sin(x - y)) + e^{-x}(y - \cos(x - y)), e^{-x}(x + \cos(x - y))\right)$$

Thus $\nabla c(1, 1) = (-\frac{4}{e}, \frac{2}{e})$. Normalizing, the required direction is thus that of $\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$.

(b) Let the flow line be denoted $\mathbf{b}(t)$, so $\mathbf{b}(0) = (1, 1)$ and $\mathbf{b}'(t) = \mathbf{F}(\mathbf{b}(t))$. The required rate is

$$\frac{d}{dt}c(\mathbf{b}(t)) = \nabla c(\mathbf{b}(t)) \cdot \mathbf{b}'(t);$$

at $t = 0$, this is $(-\frac{4}{e}, \frac{2}{e}) \cdot (3\mathbf{i} + 2\mathbf{j}) = -\frac{12}{e} + \frac{4}{e} = -\frac{8}{e}$.

(c) The acceleration is $\mathbf{b}''(0)$. Now if $\mathbf{b}(t) = (x(t), y(t))$ we have

$$\mathbf{b}'(t) = (x'(t), y'(t)) = 3x(t)\mathbf{i} + 2\mathbf{j}$$

and so $\mathbf{b}''(t) = 3x'(t)\mathbf{i} = 9x(t)\mathbf{i}$. Thus, $\mathbf{b}''(0) = 9x(0)\mathbf{i} = 9\mathbf{i}$. ◊

Problem 2

(a) *The temperature in space at the position (x, y, z) is given by the function $T(x, y, z) = x^2 + y^2 - 3z^2$. If a person is at $(x, y, z) = (0, 1, 1)$, in what direction must they go to increase the temperature as fast as possible?*

(b) *The person in (a) decides to follow the flow line of the vector field* $\mathbf{F} = \nabla T$, *again starting at* $(0,1,1)$ *at* $t = 0$. *Show that the rate of change of their temperature is given by* $\|\nabla T(0,1,1)\|^2$ *at* $t = 0$.

(c) *Describe geometrically what paths, starting at* $(0,1,1)$, *the person in (a) can take to maintain the same temperature.*

Solution.

(a) The required direction is that of the gradient; $\nabla T = 2x\mathbf{i} + 2y\mathbf{j} - 6z\mathbf{k}$, which at $(0,1,1)$ becomes $\nabla T(0,1,1) = 2\mathbf{j} - 6\mathbf{k}$.

(b) The flow line of $\mathbf{F} = \nabla T$ satisfies $\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)) = \nabla T(\mathbf{c}(t))$. By the chain rule,

$$\begin{aligned} \frac{d}{dt} T(\mathbf{c}(t)) &= \nabla T(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \\ &= \|\nabla T(\mathbf{c}(t))\|^2. \end{aligned}$$

In particular, at $t = 0$ we get $\|\nabla T(0,1,1)\|^2$.

(c) To maintain the same temperature, we must follow a curve in the level set $T = T(0,1,1) = -2$; such a curve is orthogonal to $\nabla T(0,1,1)$ at $(0,1,1)$. ◊

Part (c) of the next example assumes the students know a little linear algebra terminology.

Problem 3

(a) *Let* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *be defined by*

$$f(x,y,z) = (e^{-2xy}, x^2 - z^2 - 4x + \sin(x + y + z))$$

and let $g : \mathbb{R}^2 \to \mathbb{R}$ *be a function such that* $g(1,0) = -1$, *and* $\nabla g(1,0) = \mathbf{i} - 3\mathbf{j}$. *Calculate the gradient of* $g \circ f$ *at the point* $(0,0,0)$.

(b) *Find the equation of the tangent plane to the level set* $g \circ f = -1$ *at the point* $(0,0,0)$, *where* g *and* f *are defined in part (a).*

(c) *If* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *and* $g : \mathbb{R}^2 \to \mathbb{R}$, *show that a given point* (x_0, y_0, z_0) *is a critical point of the composition* $g \circ f$ *if and only if the range of* $\mathbf{D}f(x_0, y_0, z_0)$ *lies in the kernel of* $\mathbf{D}g(u_0, v_0)$, *where* $(u_0, v_0) = f(x_0, y_0, z_0)$.

Solution.

(a) Let $f(x,y,z) = (u(x,y,z), v(x,y,z))$ and $h(x,y,z) = g(f(x,y,z))$. By the chain rule,

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial h}{\partial y} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial y}$$

and

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial z}.$$

Thus, at $(0,0,0)$,

$$\frac{\partial h}{\partial x} = (1)(0) + (-3)(-4+1) = 9$$
$$\frac{\partial h}{\partial y} = (1)(0) + (-3)(1) = -3$$
$$\frac{\partial h}{\partial z} = (1)(0) + (-3)(1) = -3.$$

Thus, the required gradient is $9\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$.

(b) The normal is the vector $9\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$, and thus the tangent plane is

$$(x-0)(9) + (y-0)(-3) + (z-0)(-3) = 0$$

i.e.,

$$9x - 3y - 3z = 0$$

i.e.,

$$3x - y - z = 0.$$

(c) By the chain rule or (a),

$$\mathbf{D}h(x_0, y_0, z_0) = \mathbf{D}g(u_0, v_0) \circ \mathbf{D}f(x_0, y_0, z_0)$$

and so $\mathbf{D}h(x_0, y_0, z_0) = 0$ if and only if $\mathbf{D}g(u_0, v_0,) \circ \mathbf{D}f(x_0, y_0, z_0) = 0$, which means that the range of $\mathbf{D}f(x_0, y_0, z_0)$ lies in the kernel of $\mathbf{D}g(u_0, v_0)$. ◊

Problem 4 *Extremize* $f(x,y,z) = z$ *subject to the constraints*

$$x^2 + y^2 + z^2 = 1 \text{ and } x + y + z = 1.$$

Solution. To do this, we use Lagrange multipliers. Let $g_1(x, y, z) = x^2 + y^2 + z^2 - 1$ and $g_2(x, y, z) = x + y + z - 1$ so we are extremizing f subject to $g_1 = 0$ and $g_2 = 0$. By the Lagrange multiplier method, we must solve the system

$$\begin{aligned} \nabla f &= \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 &= 0 \\ g_2 &= 0 \end{aligned}$$

for $(x, y, z, \lambda_1, \lambda_2)$. The system is

$$\begin{aligned} 0 &= \lambda_1 \cdot 2x + \lambda_2 \cdot 1 \\ 0 &= \lambda_1 \cdot 2y + \lambda_2 \cdot 1 \\ 1 &= \lambda_1 \cdot 2z + \lambda_2 \cdot 1 \\ x^2 + y^2 + z^2 &= 1 \\ x + y + z &= 1 \end{aligned}$$

i.e.,

$$\begin{aligned} 2\lambda_1 x + \lambda_2 &= 0 \\ 2\lambda_1 y + \lambda_2 &= 0 \\ 2\lambda_1 z + \lambda_2 &= 1 \\ x^2 + y^2 + z^2 &= 1 \\ x + y + z &= 1. \end{aligned}$$

Thus, $2\lambda_1(x - y) = 0$ so either $\lambda_1 = 0$ or $x = y$. If $\lambda_1 = 0$ then $\lambda_2 = 0$ from the first equation and $\lambda_2 = 1$ from the third. Hence $\lambda_1 \neq 0$ and therefore $x = y$. Thus, our system becomes, with $y = x$,

$$\begin{aligned} 2\lambda_1 x + \lambda_2 &= 0 \\ 2\lambda_1 z + \lambda_2 &= 1 \\ 2x^2 + z^2 &= 1 \\ 2x + z &= 1. \end{aligned}$$

We can now solve the last two equations for z and x; $z = 1 - 2x$, so the third equation gives

$$2x^2 + (1 - 2x)^2 = 1$$

or

$$6x^2 - 4x = 0$$

and so $x = 0$ or $x = 2/3$. Thus the extrema points are

$$(0, 0, 1) \quad \text{and} \quad (2/3, 2/3, -1/3).$$

The function f has a maximum at the first point and a minimum at the second. $\Diamond$

Problem 5 *Extremize* $f(x,y,z) = x + y$ *subject to the constraints*

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad y + z = 1.$$

Solution. The Lagrange multiplier criterion states that we seek points such that $\nabla f = \lambda \nabla g + \mu \nabla h$ where $g(x,y,z) = x^2 + y^2 + z^2 - 1$ and $h(x,y,z) = y + z - 1$. Thus our system is

$$\begin{aligned} 1 &= \lambda \cdot 2x \\ 1 &= \lambda \cdot 2y + \mu \\ 0 &= \lambda \cdot 2z + \mu \\ x^2 + y^2 + z^2 &= 1 \\ y + z &= 1. \end{aligned}$$

From the second and third equation, $2\lambda(y - z) = 1$ and so from this and the first equation,

$$2\lambda(y - z - x) = 0.$$

Now the first equation shows $\lambda \neq 0$, so $z = y - x$. This, and $z = 1 - y$ gives $x = 2y - 1$. Thus $x^2 + y^2 + z^2 = 1$ becomes $(2y-1)^2 + y^2 + (1-y)^2 = 1$ *i.e.*, $6y^2 - 6y + 1 = 0$ *i.e.*, $y = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$. Thus, the two extrema are

$$\left(\pm\frac{\sqrt{3}}{3},\ \frac{1}{2} \pm \frac{\sqrt{3}}{6},\ \frac{1}{2} \mp \frac{\sqrt{3}}{6} \right). \quad \Diamond$$

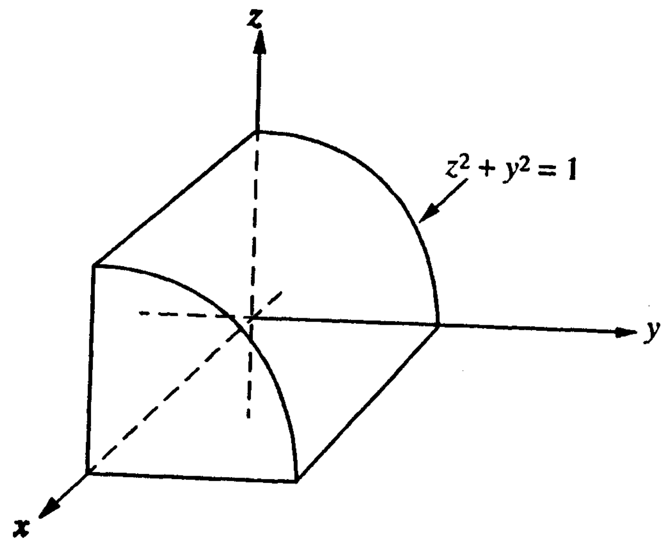
Problem 6 *Rewrite the integral*

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-y^2}} f(x,y,z)dz\,dy\,dx$$

in the order $dy\,dz\,dx$*, including a sketch of the region of integration.*

Solution. The region of integration is shown in the following figure. In the order $dy\,dz\,dx$ we would get

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-z^2}} f(x,y,z)dy\,dz\,dx. \quad \Diamond$$



The region of integration for Problem 6

Problem 7

(a) *Evaluate the following integral*

$$\int_0^2 \int_0^x \int_0^{x+y} dz\, dy\, dx.$$

(b) *Describe the region of integration for the integral in (a).*

Solution. For (a) note that

$$\begin{aligned}\int_0^2 \int_0^x \int_0^{x+y} dz\, dy\, dx &= \int_0^2 \int_0^x (x+y) dy\, dx \\ &= \int_0^2 \left(xy + \frac{y^2}{2} \right) \Big|_0^x dx \\ &= \int_0^2 \frac{3x^2}{2} dx = \frac{x^3}{2} \Big|_0^2 = 4.\end{aligned}$$

For (b), and referring to the next figure, we see that the region is above the triangle in the (xy)-plane with vertices $(0,0), (2,0)$ and $(2,2)$ and below the plane $z = x + y$.

Problem 8 *An alien force field exerted on Captain George's spaceship is given by* $\mathbf{F} = -45\mathbf{r}/r^5$. *Find the work done needed to move the ship against this field from a distance r_1 to a distance $r_2 > r_1$.*

Solution. The vector field $-\mathbf{F}$ is a gradient:

$$-\mathbf{F} = \nabla f$$

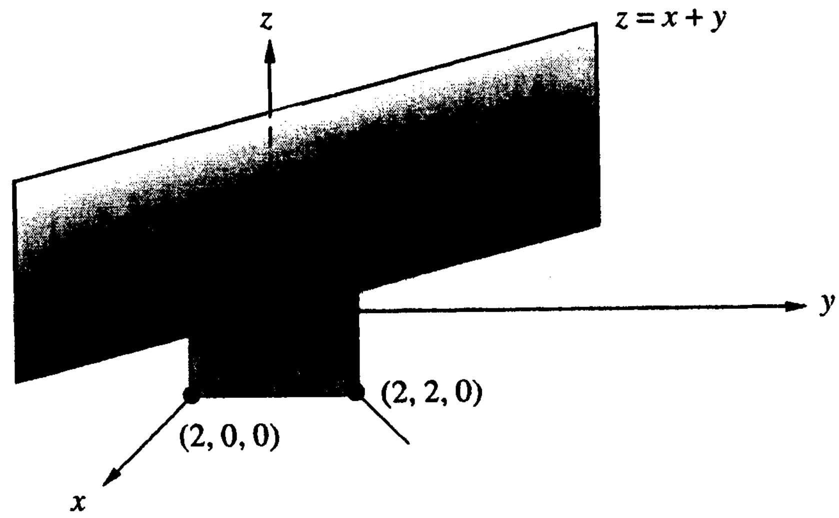


FIGURE 8.6.1. Figure for Problem 7

where $f = -15/r^3$. Thus the work done by the opposing field $-\mathbf{F}$ is the line integral

$$-\int_C \mathbf{F} \cdot d\mathbf{S} = f(\text{end}) - f(\text{start}) = \frac{15}{r_1^3} - \frac{15}{r_2^3}. \quad \diamond$$

Problem 9 *An alien force field is exerting the force $\mathbf{F} = -10\mathbf{r}/r^6$ on Captain Alice's spaceship. Find the work done by the field in moving the ship from a distance $r = 10$ to a distance $r = 9$.*

Solution. Here $\mathbf{F} = \nabla f$ where $f = 5/(2r^4)$, so the work done is

$$\begin{aligned}\int \mathbf{F} \cdot d\mathbf{S} &= f(\text{end}) - f(\text{start}) \\ &= \frac{5}{2}\left(\frac{1}{9^4} - \frac{1}{10^4}\right) = \frac{5}{2}\frac{10^4 - 9^4}{(90)^4}. \quad \diamond\end{aligned}$$

Problem 10

(a) *Let $\Phi(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$, for $0 \le u \le \pi/2$ and $\pi/2 \le v \le \pi$. Describe the parametrized surface S so obtained.*

(b) *Find the equation of the tangent plane to S at $u = \pi/4, v = 3\pi/4$.*

(c) *For a smooth function $f(x, y, z)$ and a general parametrized surface that satisfies $\Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0) \neq 0$, show that $(x_0, y_0, z_0) = \Phi(u_0, v_0)$ is a critical point of f if the following two conditions hold:*

1. $\nabla f(x_0, y_0, z_0) \cdot [\Phi_u(u_0, v_0) \times \Phi_v(u_0, v_0)] = 0$
2. $\nabla f(x_0, y_0, z_0) \cdot \sigma'(0) = 0$, *where σ is any curve of the form $\sigma(t) = \Phi(\mathbf{c}(t))$, with $\mathbf{c}(t)$ a curve in the uv plane satisfying $\mathbf{c}(0) = (u_0, v_0)$.*

Solution.

(a) Letting $u = \phi$ and $v = \theta$, we recognize

$$\Phi(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

as a portion of the sphere $x^2 + y^2 + z^2 = 1$ of radius 1; with $0 \le \phi \le \pi/2$, $\pi/2 \le \theta \le \pi$, it is in the octant with $z \ge 0, x \le 0, y \ge 0$.

(b) At $\phi = \pi/4, \theta = 3\pi/4, x = -1/2, y = 1/2, z = 1/\sqrt{2}$, so the normal, $2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ becomes $-\mathbf{i} + \mathbf{j} + \sqrt{2}\,\mathbf{k}$ and the tangent plane thus has equation

$$-\left(x + \frac{1}{2}\right) + \left(y - \frac{1}{2}\right) + \sqrt{2}\left(z - \frac{1}{2}\right) = 0;$$

that is,

$$\sqrt{2}\,z + y - x = 1 + \frac{\sqrt{2}}{2}.$$

(c) Condition 1 says that ∇f is perpendicular to the normal to the surface while 2 says that ∇f is perpendicular to the tangent plane to the surface. Hence $\nabla f(x_0, y_0, z_0) = 0$ and so (x_0, y_0, z_0) is a critical point. ◊

Problem 11 *Let* $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$. *Show that the integral of* $\mathbf{F}$ *around the circumference of the square* $[0,1] \times [0,1]$ *in the xy-plane is zero by*

(a) *direct evaluation*

(b) *showing that* $\mathbf{F}$ *is a gradient—find the function it is the gradient of, and*

(c) *using Green's Theorem.*

Solution.

(a) First we evaluate the integral along the x-axis from $x = 0$ to $x = 1$. With $y = 0$ and x as the parameter, we get

$$\int_{x=0}^{1} 2xy\,dx + x^2 dy = \int_0^1 0 \cdot dx + x^2 \cdot dy = 0$$

since $dy = 0$. Along the line $x = 1$ from $y = 0$ to $y = 1$ we get

$$\int_{y=0}^{1} 2xy\,dx + x^2 dy = 1.$$

Along the line $y = 1$ from $x = 1$ to $x = 0$ gives

$$\int_{x=1}^{0} 2x\,dx + x^2 dy = -1$$

and finally along the line $x = 0$ from $y = 1$ to 0 gives zero. Adding these gives zero.

(b) By inspection, we see that $\mathbf{F} = \nabla f$ where $f(x,y) = x^2y$; since

$$\int_C \mathbf{F} \cdot d\mathbf{S} = f(\text{end}) - f(\text{start})$$

and the curve is closed, we get zero.

(c) Green's theorem states that

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{\partial D} P dx + Q dy$$

and in this case $P = 2xy$ and $Q = x^2$, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2x = 0,$$

so we again get zero. $\Diamond$

Problem 12 *If true, **justify**, and if false, give a **counterexample**, or explain why.*

(a) *The path integral $\int_{\mathbf{c}} 2\pi\, ds$ is the surface area of a cylinder of radius 1 and height 2π where the curve is defined by $\mathbf{c} = (\cos t, \sin t, 0)$, and $0 \leq t \leq 2\pi$.*

(b) *If $f(x,y)$ is a smooth function defined on the disk $x^2 + y^2 < 1$ and has a strict minimum at the origin $(0,0)$, then the matrix of second partial derivatives of f at $(0,0)$ is positive definite.*

(c) *If $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ on the disk $x^2 + y^2 < 1$, then*

$$\int_C \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = 0,$$

where C is the circle of radius $\frac{1}{2}$ centered at the origin.

(d) *There is no vector field $\mathbf{F}$ such that $\nabla \times \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.*

(e) *The flux of a (smooth) vector field $\mathbf{F}$ out of the unit sphere $x^2 + y^2 + z^2 = 1$ equals $(4\pi/3)\,\mathrm{div}\,\mathbf{F}(P)$ for some point P inside the sphere.*

(f) *If f is a smooth function of (x,y), then there is at least one point (x_0, y_0) on the circle $x^2 + y^2 = 1$ such that $\nabla f(x_0, y_0) = k(x_0\mathbf{i} + y_0\mathbf{j})$ for some constant k.*

Solution.

(a) True; the path integral is $(2\pi)^2$ and the surface area is (circumference) $\times$ (height) $= (2\pi) \times (2\pi) = (2\pi)^2$.

(b) False, a counterexample is $f(x, y) = x^4 + y^4$.

(c) By Green's theorem,

$$\int_C \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy = \iint_D \left(-\frac{\partial}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}\frac{\partial u}{\partial y} \right) dx\, dy = 0,$$

so the result is true.

(d) If there were such an $\mathbf{F}$, then $0 = \nabla \cdot (\nabla \times \mathbf{F}) = \nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3$, a contradiction. So the statement is true, there is no such $\mathbf{F}$.

(e) True. Note that the flux is $\iiint_W \operatorname{div} \mathbf{F}\, dx\, dy\, dz$ by the divergence theorem. Now apply the mean value theorem.

(f) True, by the Lagrange multiplier theorem with the constraint $x^2 + y^2 - 1 = 0$. $\Diamond$

Part (b) of the following example assumes students know about eigenvalues.

Problem 13 *If true, **justify** and, if false, give a **counterexample**, or explain why.*

(a) *If $f(x, y, z)$ is a smooth function, then the flux of ∇f out of the sphere $x^2 + y^2 + z^2 = 1$ is zero.*

(b) *If $f(x, y)$ is a smooth function defined on the disk $x^2 + y^2 < 1$, if $f(0, 0) = 0$ and if $f(x, y) > 0$ for all $(x, y) \neq 0$, then the matrix of second partial derivatives of f evaluated at $(0, 0)$ has real eigenvalues λ_1 and λ_2 that satisfy $\lambda_1 > 0$ and $\lambda_2 > 0$.*

(c) *There exists a vector field $\mathbf{F}$ that satisfies $\nabla \times \mathbf{F} = x\mathbf{i}$.*

(d) *The line integral of a smooth vector field $\mathbf{F}$ around the disk $x^2 + y^2 = r^2, z = 0$ equals $\pi r^2 [(\nabla \times \mathbf{F})(0, 0, 0)] \cdot \mathbf{k}$.*

(e) *The velocity field of a fluid is given by $\mathbf{F} = y\mathbf{j} + 2z\mathbf{k}$. The rate of flow of fluid out of the sphere $x^2 + y^2 + z^2 = 1$ is 4π.*

Solution.

(a) False, for example, if $f = \frac{1}{2}(x^2 + y^2 + z^2)$, then $\nabla f = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ has *positive* flux out of the sphere.

(b) False; $f(x, y) = x^4 + y^4$ would have $\lambda_1 = 0 = \lambda_2$.

(c) False since $0 = \nabla \cdot (\nabla \times \mathbf{F}) \neq 1 = \mathrm{div}(x\mathbf{i})$.

(d) False; it is $\pi r^2 [\nabla \times \mathbf{F}(P)] \cdot \mathbf{k}$ at *some* point in the disk that need not be the origin.

(e) The divergence $\mathbf{F}$ is 3 and so by the divergence theorem, the rate of flow is $3 \times$ (volume) $= 3 \times \frac{4\pi}{3} = 4\pi$, so it is true. ◊

Problem 14 *Let W be the region in space under the graph of*

$$f(x, y) = (\cos y)\exp(1 - \cos 2x) + xy$$

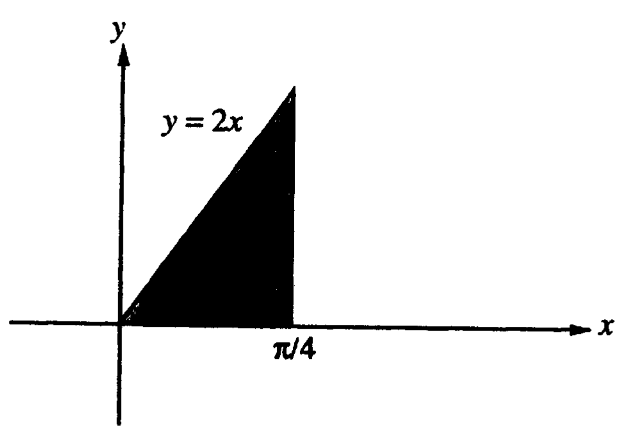
over the region in the xy plane bounded by the line $y = 2x$, the x axis, and the line $x = \pi/4$.

(a) *Find the volume of W.*

(b) *Let $\mathbf{F} = 5x\mathbf{i} + 5y\mathbf{j} + 5z\mathbf{k}$ be the velocity field of a fluid in space. Calculate the rate at which fluid is leaving the region W in part (a).*

Solution.

(a) The region in the xy-plane is as shown in the following figure.



The region for Problem 14

The volume of W is

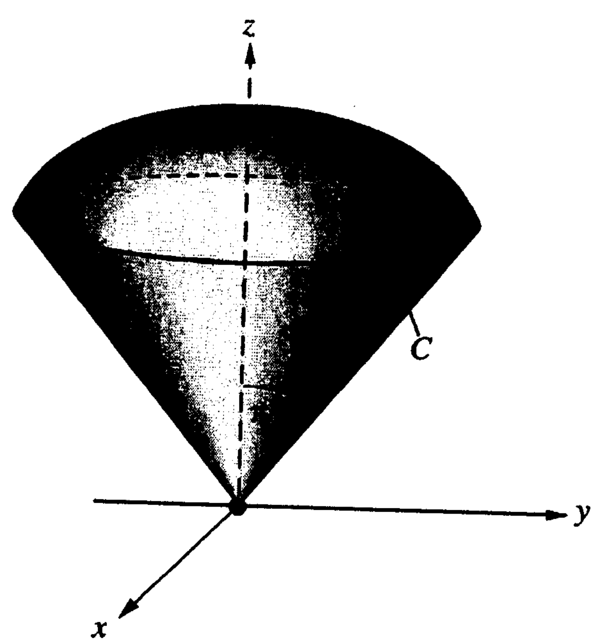
$$\iint_D f(x,y)dy\,dx = \int_0^{\pi/4}\int_0^{2x}[(\cos y)\exp(1-\cos 2x)+xy]dy\,dx$$
$$= \int_0^{\pi/4}\left[(\sin 2x)\exp(1-\cos 2x)+x\cdot\frac{(2x)^2}{2}\right]dx$$
$$= \left[\frac{1}{2}\exp(1-\cos 2x)+\frac{x^4}{2}\right]\Big|_0^{\pi/4}$$
$$= \frac{1}{2}(e-1)+\frac{\pi^4}{512}.$$

(b) By the divergence theorem, the flux is

$$\iiint_W \text{div}\,\mathbf{F}\,dx\,dy\,dz = 5\times 3\iiint_W dx\,dy\,dz,$$

so the flux is $15\left[\frac{1}{2}(e-1)+\frac{\pi^4}{512}\right]$ by (a). ◊

Problem 15 *Let S be the spherical cap formed by cutting the sphere $x^2 + y^2 + z^2 = 1$ with a cone having a vertex angle $\pi/6$ and with the vertex at the center of the sphere.*



The figure for Problem 15

(a) *Find the area of S.*

(b) *Let C denote the boundary of the surface (the cap) S considered in part (a) and let $\mathbf{F} = (z - y)\mathbf{i} + y\mathbf{k}$. Calculate the line integral of $\mathbf{F}$ around the curve C using a chosen orientation on C. Do this calculation* **both** *directly* **and** *by using Stokes' theorem.*

Solution.

(a) The area element on a sphere is $\rho^2 \sin\phi \, d\theta \, d\phi$. Here, $\rho = 1$ and the area is

$$\begin{aligned} \int_{\phi=0}^{\pi/6} \int_{\theta=0}^{2\pi} \sin\phi \, d\theta \, d\phi &= -2\pi \cos\phi \Big|_0^{\pi/6} \\ &= 2\pi(1 - \cos(\pi/6)) \\ &= 2\pi\left(1 - \frac{\sqrt{3}}{2}\right) = \pi(2 - \sqrt{3}). \end{aligned}$$

(b) We integrate around C counterclockwise viewed from above, so the corresponding orientation of S is *outward*. To compute $\int_C \mathbf{F} \cdot d\mathbf{S}$ directly, we parametrize C as follows. It is a circle of radius $\sin(\pi/6) = 1/2$ and at height $z = \cos(\pi/6) = \sqrt{3}/2$, so a parametrization is

$$\mathbf{c}(t) = \frac{1}{2}\cos t\,\mathbf{i} + \frac{1}{2}\sin t\,\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}.$$

Thus,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{S} &= \int_{t=0}^{2\pi} (z - y)dx + y\,dz \\ &= \int_0^{2\pi} \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\sin t\right) \cdot \frac{1}{2}(-\sin t)dt \\ &= \frac{1}{4}\int_0^{2\pi} \sin^2 t \, dt = \frac{\pi}{4}. \end{aligned}$$

On the other hand, by Stokes' theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

and

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z - y & 0 & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$

so

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_S (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot [x\mathbf{i} + y\mathbf{j} + z\mathbf{k}] dS$$
$$= \iint_S (x + y + z) dS = \iint_S z dS,$$

since $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is a unit outward normal and the x and y integrals vanish by symmetry. But, as in (a),

$$\iint_S z\, dS = \int_0^{\pi/6} \int_0^{2\pi} \cos\phi \sin\phi \, d\theta \, d\phi$$
$$= 2\pi \cdot \left. \frac{\sin^2\phi}{2} \right|_0^{\pi/6} = 2\pi \cdot \frac{1}{2} \sin^2(\pi/6) = \frac{\pi}{4}.$$

An alternative (perhaps cleverer) way to do (b) is to use Stokes' theorem on the flat disk S_1 defined by $x^2 + y^2 \leq 1/2$ and $z = \sqrt{3}/2$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$
$$= \iint_{S_1} (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \mathbf{k} \, dx \, dy$$
$$= \iint_{S_1} dx \, dy = \text{area}(S_1) = \pi/4.$$

Problem 16 *Let C be the circle $x^2 + y^2 = 1, z = 0$, and let*

$$\mathbf{F}(x, y, z) = [x^2y^3 + y - \cos(x^3)]\mathbf{i} + [x^3y^2 + \sin(y^3) + x]\mathbf{j} + z\mathbf{k}.$$

Calculate the line integral $\int_C \mathbf{F} \cdot d\mathbf{S}$.

Solution. By Stokes' theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dx \, dy$$

where S is the disk $x^2 + y^2 \leq 1, z = 0$. The $\mathbf{k}$-component of $\nabla \times \mathbf{F}$ is

$$\frac{\partial}{\partial x}(x^3y^2 + \sin(y^3) + x) - \frac{\partial}{\partial y}(x^2y^3 + y - \cos(x^3))$$
$$= 3x^2y^2 + 1 - 3x^2y^2 - 1 = 0.$$

Hence the integral is zero. ◊