M427L (55200), Sample Final Exam Questions

Below are some sample final exam questions. Please note that the intent of these is to help prepare for the exam, and that actual exam questions will not merely be modifications of these problems. Consult your HW, quizzes, lecture notes, and book for additional sources of material to review. Finally, to obtain the full effect of an exam please complete these problems under time-pressure (120-150 minutes).

1.

a) The concentration of a toxic chemical in the xy-plane at position (x, y) is given by $c(x, y) = e^{-x}(xy + 3 - \sin(x - y))$. If an environmental worker is at $(x, y) = (1, 1)$, in what direction must she go to decrease the concentration as fast as possible?

b) The environmental worker decides to follow the flow line of the vector field defined by $\boldsymbol{F}(x, y) = 3x\boldsymbol{i} + 2\boldsymbol{j}$, again starting at $(1, 1)$. How fast is the concentration of the chemical changing as the worker starts along the flow line, as a function of time t along the flow line ($t = 0$ corresponds to the starting position)?

c) What is the acceleration of the worker following the path in (b) at $t = 0$?

2.

a) The temperature in space at the position (x, y, z) is given by the function $T(x, y, z) = x^2 + y^2 - 3z^2$. If a person is at $(x, y, z) = (0, 1, 1)$, in what direction must they go to increase the temperature as fast as possible?

b) The person in (a) decides to follow the flow line of the vector field $\boldsymbol{F} = \nabla T$, again starting at $(0, 1, 1)$ at $t = 0$. Show that the rate of change of their temperature is given by $\|\nabla T(0, 1, 1)\|^2$ at $t = 0$.

c) Describe geometrically what paths, starting at $(0, 1, 1)$, the person in (a) can take to maintain the same temperature.

3.

a) Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x, y, z) = \left(e^{-2xy}, x^2 - z^2 - 4x + \sin(x + y + z)\right)$$

and let $g\colon \mathbb{R}^2 \to \mathbb{R}$ be a function such that $g(1, 0) = -1$, and $\nabla g(1, 0) = \boldsymbol{i} - 3\boldsymbol{j}$. Calculate the gradient of $g \circ f$ at the point $(0, 0, 0)$.

b) Find the equation of the tangent plane to the level set $g \circ f = -1$ at the point $(0, 0, 0)$, where g and f are defined in part (a).

c) [not included]

4. Extremize $f(x, y, z) = z$ subject to the constraints

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x + y + z = 1.$$

5. Extremize $f(x, y, z) = x + y$ subject to the constraints

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad y + z = 1.$$

6. Rewrite the integral

$$\int_0^1 \int_0^1 \int_0^{\sqrt{1-y^2}} f(x, y, z)\, dz dy dx$$

in the order $dydzdx$, including a sketch of the region of integration.

7.

a) Evaluate the following integral

$$\int_0^2 \int_0^x \int_0^{x+y} dz dy dx.$$

b) Describe the region of integration for the integral in (a).

8. An alien force field exerted on Captain George's spaceship is given by $\boldsymbol{F} = -45\boldsymbol{r}/r^5$. Find the work done needed to move the ship against this field from a distance r_1 to a distance $r_2 > r_1$.

9. An alien force field is exerting the force $\boldsymbol{F} = -10\boldsymbol{r}/r^6$ on Captain Alice's spaceship. Find the work done by the field in moving the ship from a distance $r = 10$ to a distance $r = 9$.

10.

a) Let $\boldsymbol{\Phi}(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$, for $0 \le u \le \pi/2$ and $\pi/2 \le v \le \pi$. Describe the parametrized surface so obtained.

b) Find the equation of the tangent plane to S at $u = \pi/4$, $v = 3\pi/4$.

c) For a smooth function $f(x, y, z)$ and a general parametrized surface that satisfies $\boldsymbol{\Phi}_u(u_0, v_0) \times \boldsymbol{\Phi}_v(u_0, v_0) \neq \boldsymbol{0}$, show that $(x_0, y_0, z_0) = \boldsymbol{\Phi}(u_0, v_0)$ is a critical point of f if the following two conditions hold:

1. $\nabla f(x_0, y_0, z_0) \cdot [\boldsymbol{\Phi}_u(u_0, v_0) \times \boldsymbol{\Phi}_v(u_0, v_0)] = 0$

2. $\nabla f(x_0, y_0, z_0) \cdot \sigma'(0) = 0$, where σ is any curve of the form $\sigma(t) = \boldsymbol{\Phi}(\boldsymbol{c}(t))$, with $\boldsymbol{c}(t)$ a curve in the uv-plane satisfying $\boldsymbol{c}(0) = (u_0, v_0)$.

11. Let $\boldsymbol{F}(x, y) = 2xy\boldsymbol{i} + x^2\boldsymbol{j}$. Show that the integral of $\boldsymbol{F}$ around the circumference of the square $[0, 1] \times [0, 1]$ in the xy-plane is zero by

a) direct evaluation

b) showing that $\boldsymbol{F}$ is a gradient—find the function it is a gradient of, and

c) using Green's theorem.

12. If true, **justify**, and if false, give a **counterexample**, or explain why.

a) The path integral $\int_{\boldsymbol{c}} 2\pi\, ds$ is the surface area of a cylinder of radius 1 and height 2π where the curve is defined by $\boldsymbol{c} = (\cos t, \sin t, 0)$, and $0 \leq t \leq 2\pi$.

b) If $f(x, y)$ is a smooth function defined on the disk $x^2 + y^2 < 1$ and has a strict minimum at the origin $(0, 0)$, then the matrix of second partial derivatives of f at $(0, 0)$ is positive definite.

c) If $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ on the disk $x^2 + y^2 < 1$, then

$$\int_C \frac{\partial u}{\partial y}\, dx - \frac{\partial u}{\partial x}\, dy = 0,$$

where C is the circle of radius $\frac{1}{2}$ centered at the origin.

d) There is no vector field $\boldsymbol{F}$ such that $\nabla \times \boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

e) [not included]

f) If f is a smooth function of (x, y), then there is at least one point (x_0, y_0) on the circle $x^2 + y^2 = 1$ such that $\nabla f(x_0, y_0) = k(x_0\boldsymbol{i} + y_0\boldsymbol{j})$ for some constant k.

13. If true, **justify**, and if false, give a **counterexample**, or explain why.

a) If f is a smooth function, then the flow of ∇f out of the sphere $x^2 + y^2 + z^2 = 1$ is zero.

b) [not included]

c) There is a vector field $\boldsymbol{F}$ that satisfies $\nabla \times \boldsymbol{F} = x\boldsymbol{i}$.

d) The line integral of a smooth vector field $\boldsymbol{F}$ around the disk $x^2 + y^2 = r^2$, $z = 0$ equals $\pi r^2 [(\nabla \times \boldsymbol{F})(0, 0, 0)] \cdot \boldsymbol{k}$.

e) The velocity field of a fluid is given by $\boldsymbol{F} = y\boldsymbol{j} + 2z\boldsymbol{k}$. The rate of flow of fluid out of the sphere $x^2 + y^2 + z^2 = 1$ is 4π.

14. Let W be the region in space under the graph of

$$f(x, y) = (\cos y) \exp(1 - \cos 2x) + xy$$

over the region in the xy-plane bounded by the line $y = 2x$, the x-axis, and the line $x = \pi/4$.

a) Find the volume of W.

b) Let $\boldsymbol{F} = 5x\boldsymbol{i} + 5y\boldsymbol{j} + 5z\boldsymbol{k}$ be the velocity field of a fluid in space. Calculate the rate at which fluid is leaving the region W in part (a).

15. Let S be the spherical cap formed by cutting the sphere $x^2 + y^2 + z^2 = 1$ with a cone having a vertex angle $\pi/6$ and with the vertex at the center of the sphere.

a) Find the area of S.

b) Let C denote the boundary of the surface (the cap) S considered in part (a) and let $\boldsymbol{F} = (z - y)\boldsymbol{i} + y\boldsymbol{k}$. Calculate the line integral of $\boldsymbol{F}$ around the curve C using a chosen orientation on C. Do this calculation **both** directly **and** by using Stokes' theorem.

16. Let C be the circle $x^2 + y^2 = 1$, $z = 0$, and let

$$\boldsymbol{F}(x, y, z) = \left[x^2 y^3 + y - \cos\left(x^3\right)\right]\boldsymbol{i} + \left[x^3 y^2 + \sin\left(y^3\right) + x\right]\boldsymbol{j} + z\boldsymbol{k}.$$

Calculate the line integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{s}$.