KINETIC AND MEAN-FIELD GAME MODELS OF INFORMATION PROPAGATION

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INTRODUCTION:

- VAST BODY OF LITERATURE DEDICATED TO STUDY OF INFORMATION FLOWS THROUGH NETWORKS
- "INFORMATION" $\in$ {WEALTH, OPINIONS, SPINS, DISEASE, ...}
- RECENTLY POPULARIZED BY D. ALDOUS AS "FINITE-MARKOV INFORMATION-EXCHANGE (FMIE) PROCESSES".

- COMPONENTS OF MODEL

  ⇝ AGENTS $i \in G$ (VERTICES)

  ⇝ STRENGTH OF RELATIONSHIP $\lambda_{ij} \geq 0$ (EDGES)

  ⇝ INFORMATION STATE $\theta_i \in \Theta$, WHICH WE CALL "TYPE OF AGENT i".

  (VALUE AT VERTEX i)

- DYNAMICS GOVERNED BY RANDOM MATCHING OF AGENTS WITH RATES λ_{ij}.

  (CAN BY GENERALIZED TO N-ARY MATCHINGS.)

SCHEMATIC:

NETWORK GEOMETRY $(\{\lambda_{ij}\}_{i,j \in G})$

INTERACTION LAWS / UPDATE RULES (INCLUDES CONTROL POLICIES)

INITIAL CONDITIONS FOR DIST. OF TYPES IN POPULATION, FORCING, ETC.

- WE CONSIDER $|G| \to \infty$, $\lambda_{ij} = \lambda$ CONST. $\forall i,j \in G$

  $\rightsquigarrow$ MEAN-FIELD.

LET $\mu(t, d\theta) = |\{i \in G : \theta_i(t) \in d\theta\}|$ WHERE $|G| = 1$.

DIST. OF TYPES IN POPULATION $\mu_t(t, d\theta)$ $\Longleftrightarrow$ LAW OF ANY INDIVIDUAL AGENT'S TYPE.

- KAC MASTER EQN. (N-ARY INTERACTIONS)

$$\partial_t \int_{\mathbb{R}} \mu(d\theta_1)\, \phi(\theta_1) + 2\lambda \int_{\mathbb{R}} \mu(d\theta_1)\, \phi(\theta_1)$$

TYPE OF i AFTER INTERACTION.

$$= \lambda \int_{\mathbb{R}^N} \mu(d\theta_1) \cdots \mu(d\theta_N)\, \mathbb{E}\left[\theta_1'(\theta_1, \dots, \theta_N)\right]$$

AVE W.R.T. RANDOMNESS IN INTERACTION.

- REDUCES TO MAXWELL-TYPE MODEL FOR PARTICULAR INSTANCES OF INTERACTION LAWS, LIKE

$$\theta_i' = \sum_{j=1}^{N} A_{ij}\,\theta_j \,, \qquad i \in \{1, \dots, N\}.$$

A_{ij} ↰ RANDOM.

EX. (INFORMATION AGGREGATION)

$$A_{ij} \equiv 1 \quad \rightsquigarrow \quad \theta_i' = \sum_{j=1}^{N} \theta_j \,.$$

- DUFFIE-GIROUX-MANSO (AMER. ECON. JOUR., 2010).

$\rightsquigarrow$ PRICING OF FINANCIAL ASSETS IN OTC MARKETS.

$\rightsquigarrow$ BAYESIAN INTERPRETATION: PERFECT LEARNING.

NON TRIVIAL DIST. FOR A_{ij} $\Rightarrow$ IMPERFECT / BIASED LEARNING.

MANY OTHER EXAMPLES HAVE BEEN CONSIDERED

(CONSENSUS - SEEKING: $A_{ij} = \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix}$, $\lambda \in (0,1)$ RANDOM

RANDOMIZED PUBLIC GOODS GAMES:

$$A_{ij} = \begin{pmatrix} (1-\frac{1}{2}\lambda)K & \frac{1}{2}\lambda K \\ \frac{1}{2}\lambda K & (1-\frac{1}{2}\lambda)K \end{pmatrix}, \quad \begin{matrix} \lambda \in [0,1] \\ K \in [0,\infty). \end{matrix}$$

- BOBYLEV - CERCIGNANI - GAMBA (2009), BOBYLEV - WINDFALL (2011)

WEALTH DISTRIBUTION, OPINION DYNAMICS:

- TOSCANI, MATTHES, DUERING, PARESCHI AND MANY OTHERS...

ASYMPTOTIC BEHAVIOR → STATIONARY STATES

→ SELF-SIMILARITY, UNDER DYNAMIC SCALING

→ CONVERGENCE RATES

IN PARTICULAR, INTERPLAY BETWEEN INTERACTIONS AND INITIAL DATA LEADS TO INTERESTING SELF-SIMILAR STATES:

$$\partial_t \mu + \lambda \mu = \mathbb{E}\left[\mu(A_1 \cdot) * \cdots * \mu(A_N \cdot)\right](d\theta)$$

RANDOM MULTIPLIERS.

⇝ UNDER RESCALING, μ CONVERGES TO A SCALED MIXTURE OF STABLE LAWS (HEAVY-TAILED)

(CAN BE THOUGHT OF AS THE DIST. OF A RANDOM SUM OF R.V.'S.)

- Bobylev – Cercignani – Gamba (2009) $\rightsquigarrow$ Analytical
- Bassetti – Ladelli (2011) $\rightsquigarrow$ Probabilistic.

Applications of theory to particular models of interest

$\rightarrow$ Work in progress.

For now, will discuss a particular instance of a kinetic model which appears naturally in a mean-field game ...

PART II:

A MEAN-FIELD GAME MODEL OF INFORMATION PROPAGATION ...

BACKGROUND:

- AGENTS $\neq$ PARTICLES
- EACH AGENT CHOOSES A (SELFISH) STRATEGY TO MAXIMIZE/MINIMIZE THEIR INDIVIDUAL UTILITY / COST, GIVEN PRESENT STATE OF OTHERS.

$\rightsquigarrow$ OPTIMAL CONTROL OR STOPPING.

- SOLUTIONS ARE NASH EQUILIBRIA (TYPICALLY NOT UNIQUE!)

MODEL SETUP:

RECALL INFORMATION AGGREGATION (BINARY)

PRIOR TYPES $\left\{ \begin{array}{l} \theta_1 \\ \theta_2 \end{array} \right.$ $\rightarrow$ $\left. \begin{array}{l} \theta_1' = \theta_1 + \theta_2 \\ \theta_2' = \theta_1 + \theta_2 \end{array} \right\}$ POSTERIOR TYPES.

BAYESIAN FRAMEWORK:

- DUFFIE - GIROUX - MANSO (2010)

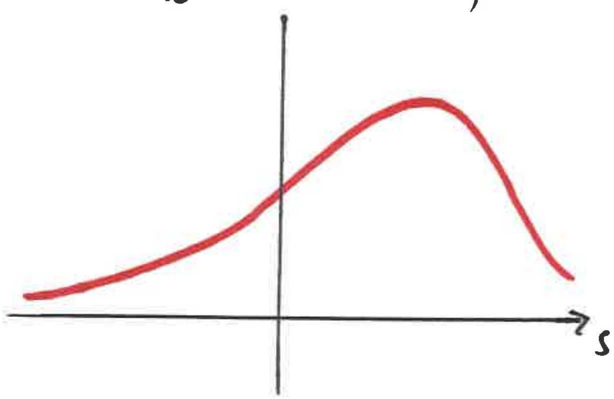
CONSIDER A GOOD WITH VALUE $X \in \{H, L\}$

r.v. (X), HIGH (H), LOW (L).

ASSUME $P(X = H) = P(X = L) = \frac{1}{2}$.

- X UNKNOWN.
- SIGNALS $\{S_k\}_{k=1}^{\infty}$: CONDITIONALLY I.I.D. R.V.'S GIVEN X
- S_k CORRELATED TO X (INFORMATIVE)

$P(s_k \in ds \mid X = H)$



$P(s_k \in ds \mid X = L)$

LOG-LIKELIHOOD RATIO GIVEN $\{s_1, \dots, s_m\}$:

$$\log \frac{P(X=H \mid s_1, \dots, s_m)}{P(X=L \mid s_1, \dots, s_m)} = \underbrace{\log \frac{P(X=H)}{P(X=L)}}_{=0} + \underbrace{\log \frac{P(s_1, \dots, s_m \mid X=H)}{P(s_1, \dots, s_m \mid X=L)}}_{:= \theta(s_1, \dots, s_m)}$$

TYPE OF $\{s_1, \dots, s_m\}$.

LAW OF LARGE NUMBERS

$$\Rightarrow \theta(s_1, \dots, s_m) = \sum_{k=1}^{m} \theta(s_k) \xrightarrow{m \to \infty} \begin{cases} +\infty, & X=H \\ -\infty, & X=L \end{cases} \quad \text{(a.s.)}$$

DYNAMICS OF ONE AGENT:

- INITIALLY, AGENT i GIVEN DISTINCT SUBSET $S_i \subset S = \{s_1, s_2, s_3, \dots\}$.

- AGENTS IN MARKET RANDOMLY MATCHED ACCORDING TO POISSON PROCESS, w/ COMMON RATE λ ACROSS AGENTS.

- UPON INTERACTING, AGENTS SHARE SIGNAL SETS:

$$\theta_i(t) = \theta_i(t-) + \theta_j(t-)$$
$$\theta_j(t) = \theta_i(t-) + \theta_j(t-).$$

WHERE

$\theta_i(t) := \theta(S_i(t))$. <u>TYPE</u> OF AGENT i AT $t \geq 0$.

- INFORMATION ACQUISITION IS COSTLY — AGENTS LEAVE MARKET WHEN SUFFICIENTLY INFORMED.

COST FOR INDIVIDUAL AGENT:

$$\underbrace{C_i(\tau)}_{\substack{\text{STOPPING}\\ \text{TIME}}} \doteq \mathbb{E}\left[\int_0^\tau \underbrace{e^{-\gamma s}}_{\substack{\text{DISCOUNTING}\\ \text{W/ RATE } \gamma \geq 0}} \underbrace{c}_{\substack{\text{RUNNING}\\ \text{COST}}}\, ds + e^{-\gamma \tau} \underbrace{\left(p_i(\tau) - \mathbb{1}_{\{X=H\}} \right)^2}_{\substack{\text{QUADRATIC EXIT COST}\\ \text{(NOT KNOWN AT EXIT!)}}} \right]$$

WHERE $p_i(t) = P(X = H \mid \mathcal{S}_i(t)) \in [0,1]$.

CAN BE WRITTEN AS

$$C_i(\tau) = \mathbb{E}\left[\int_0^\tau e^{-\gamma s} c\, ds + e^{-\gamma \tau} g(\theta_i(\tau)) \right]$$

WITH $g(\theta) = \dfrac{e^\theta}{(1+e^\theta)^2}$, $\theta \in (-\infty, \infty)$.

- CHOOSE τ TO MINIMIZE C_i ⇝ OPTIMAL STOPPING.

- NOTE: COST FUNCTION IS GIVEN BY EXPECTATION UNDER TOTAL PROBABILITY (I.E., NOT CONDITIONAL ON X).

$$\mathbb{E}\left[\left(p_i(\tau) - \mathbb{1}_{\{X=H\}}\right)^2\right]$$

$$= \mathbb{E}\Big[(p_i(\tau) - 0)^2 \underbrace{(1-p_i(\tau))}_{= P(X \neq H \mid \text{INFO. TO TIME } \tau]} + (p_i(\tau) - 1)^2 \underbrace{(p_i(\tau))}_{= P(X = H \mid \text{INFO. TO TIME } t)}$$

$$= \mathbb{E}\Big[\underbrace{p_i(\tau)(1 - p_i(\tau))}_{= g(\theta_i(\tau))}\Big].$$

⇝ COST PENALIZES "FENCE-SITTING," NOT WRONG ANSWER GIVEN X.

- MEAN-FIELD: $\mu^X(t, d\theta)$ = PROPORTION OF AGENTS ACTIVE IN MARKET AT TIME $t \geq 0$ WITH TYPE IN $d\theta$, CONDITIONAL ON X.

- $\theta_i(t)$ IS A PURE-JUMP (COMPOUND POISSON) MARKOV PROCESS W/ JUMP SIZE DISTRIBUTION $\mu^X(t, d\theta)$, GIVEN X.

$\rightsquigarrow$ HAS GENERATOR

$$\mathcal{L}_{\mu^X(t)} \underbrace{\varphi(\theta)}_{\text{TEST FCN.}} = \lambda \int_{\mathbb{R}} \left(\varphi(\theta+\eta) - \varphi(\theta)\right) \mu^X(t, d\eta).$$

- UNCONDITIONAL ON X, $\theta_i(t)$ HAS GENERATOR

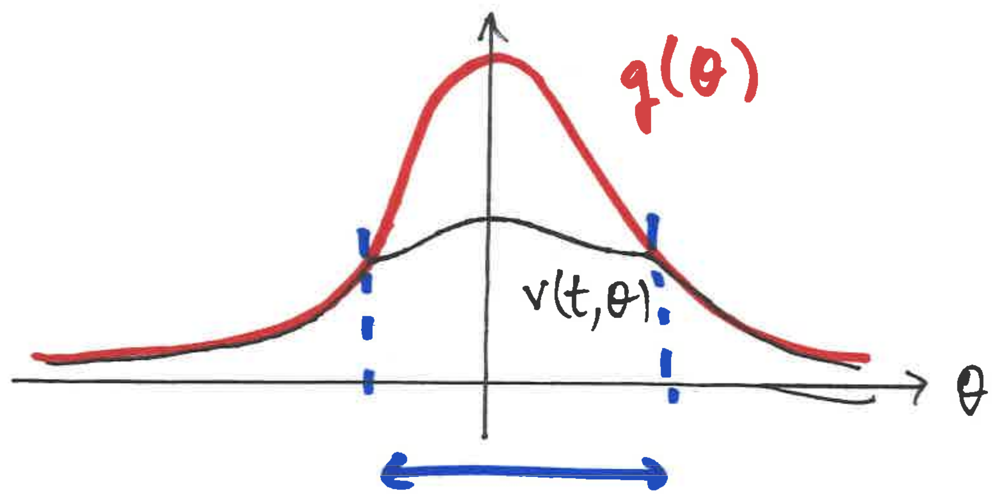
$$\overline{\mathcal{L}}_t = \frac{1}{2} \mathcal{L}_{\mu^H(t)} + \frac{1}{2} \mathcal{L}_{\mu^L(t)}.$$

OBSTACLE PROBLEM DETERMINES STOPPING REGION:

$$\max\left\{ \partial_t v - \bar{\mathcal{L}}_t v + \gamma v - c,\ v - g \right\} = 0$$

OBSTACLE

WHERE WE SOLVE FOR VALUE FUNCTION $v(t,\theta)$.



$\mathcal{R}_t \doteq$ CONTINUATION REGION $= \operatorname{supp}(v-g)$.

- Stopping region at time $t \geq 0$ is $\mathcal{R}_t^c$.

FORWARD KOLMOGOROV EQN. DETERMINES EVOLUTION OF MEAN-FIELD:

IF $P_i^X(t, d\theta)$ = DIST. OF $\theta_i(t)$, CONDITIONALLY ON X

$$\partial_t P_i^X = \mathcal{L}^{\dagger}_{\mu^X(t)} P_i^X \quad \text{IN } \mathcal{R}_t \quad \xrightarrow{\text{LLN}} \quad \boxed{\partial_t \mu^X = \mathcal{L}^{\dagger}_{\mu^X} \mu^X \quad \text{IN } \mathcal{R}_t \quad \text{w/ supp}(\mu^X) \subset \mathcal{R}_t}$$

$$\text{w/ supp}(P_i^X) \subset \mathcal{R}_t$$

- THIS IS A KINETIC EQN. ON A BOUNDED DOMAIN SINCE

$$\partial_t \mu = \mathcal{L}^{\dagger}_{\mu} \mu = \lambda \left[\mu * \mu - \mu \left(\int_{\mathcal{R}_t} \mu \right) \right] \quad \text{ON } \mathcal{R}_t .$$

To summarize:

INIT. DATA $\begin{cases} \mu^H(0, d\theta) \\ \mu^L(0, d\theta) \end{cases}$

$$\mathcal{R}_t \longmapsto \underbrace{\begin{cases} \mu^H(t, d\theta) \\ \mu^L(t, d\theta) \end{cases}}_{\text{FORWARD EQN. (KINETIC)}} \longmapsto \underbrace{\begin{cases} \mathcal{L}_{\mu^H(t)} \\ \mathcal{L}_{\mu^L(t)} \end{cases} \to \bar{\mathcal{L}}_t}_{\text{AVERAGING OVER VALUE OF } X.} \longmapsto \mathcal{R}^*_t$$

OBSTACLE g

⇝ NASH / MFG EQUILIBRIA ARE FIXED POINTS OF THIS MAP!

RELATED LITERATURE:

PREVIOUS WORK ADDRESSES OPTIMAL CONTROL OF AGENTS' MATCHING RATES.

- DUFFIE-MALAMUD-MANSO (ECONOMETRICA, 2010)
- ALDOUS (ARXIV, '10), DURRETT-CHATTERJEE (ANN. APPL. PROB, '10).

⇝ ROLE OF NETWORK GEOMETRY.

- LASRY-LIONS, GUÉANT AND CO-AUTHORS (2006 –)

$$\begin{cases} -\nu \Delta u + H(x, \nabla u) + \lambda = V(x,m) & \leftarrow \text{BACKWARD EQN. (HJB)} \\ -\nu \Delta m - \operatorname{div}\left(\frac{\partial H}{\partial p}(x, \nabla u)\, m\right) = 0 & \leftarrow \text{FORWARD EQN. (MEAN-FIELD).} \end{cases}$$

⇝ SOLUTIONS ARE NASH/MFG EQUILIBRIA.

STATIONARY PROBLEM:

- AGENT i REPLACED W/ NEW AGENT AFTER TIME $T_i \sim \mathrm{Exp}(\beta)$. NEW AGENT HAS INITIAL TYPE DISTRIBUTED ACCORDING TO INPUT MEASURES $\pi^X(d\theta)$, CONDITIONAL ON X.

- ASSUME $\pi^H(d\theta)$ SYMMETRIC TO $\pi^L(d\theta)$.

DEF. (NASH/MFG EQUILIBRIUM) $(\mathcal{R}, \mu^H)$ S.T. $\mu^H \geq 0$, $\mathrm{supp}(\mu^H) \subset \mathcal{R}$,

$$\max\left\{ -\bar{\mathcal{L}}v + (\gamma+\beta)v - (c+\beta g),\ v-g \right\} = 0 \longrightarrow \text{OBSTACLE PROBLEM}$$

$$\rightsquigarrow \mathcal{R} = \mathrm{supp}(v-g).$$

$$0 = \lambda\left(\mu^H * \mu^H - \mu^H\left(\int_{\mathcal{R}} \mu^H\right)\right) + \beta\left(\pi^H - \mu^H\right) \text{ IN } \mathcal{R} \longrightarrow \text{FORWARD EQN.}$$

- TRIVIAL NASH / MFG EQUILIBRIUM

$$(\mathcal{R}, \mu^H) = (\{\phi\}, 0).$$

- NONTRIVIAL EQUILIBRIA?

DEPENDS ON RATES: λ, β, γ

INTERACTION REPLACEMENT DISCOUNTING

AND COSTS: c, $g(\theta)$

RUNNING COST EXIT COST

AND INITIAL INFO.: π^H

INPUT MEAS. COND. ON $X=H$.

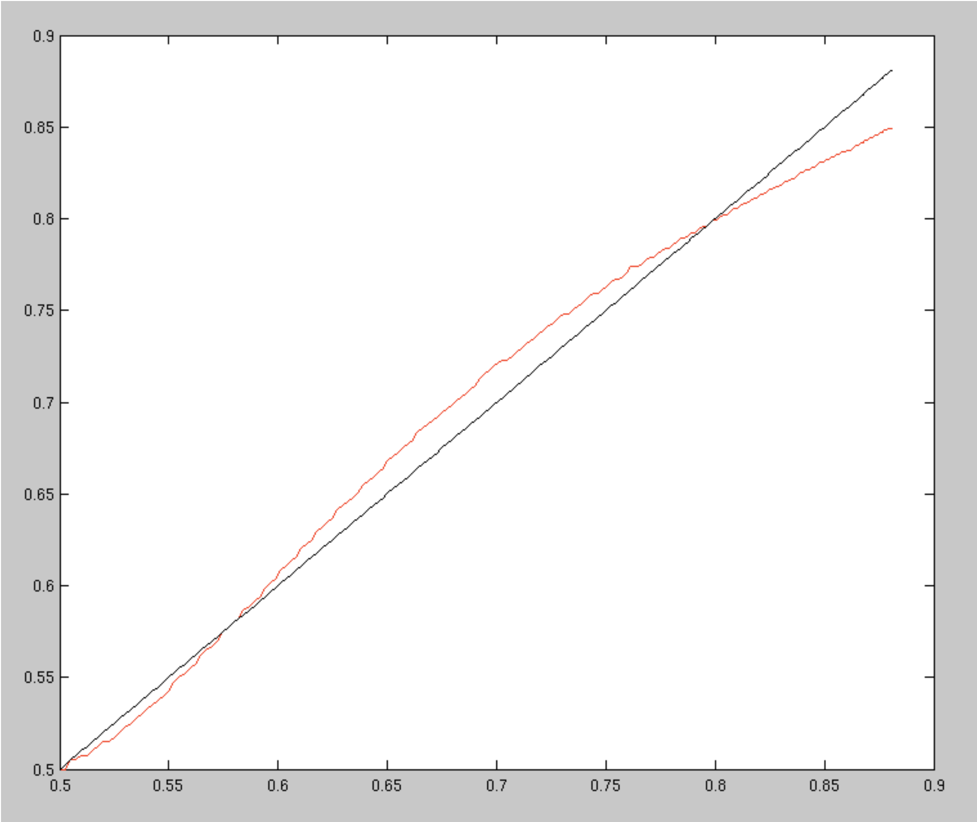
⤳ YES. (INDICATED BY NUMERICS).

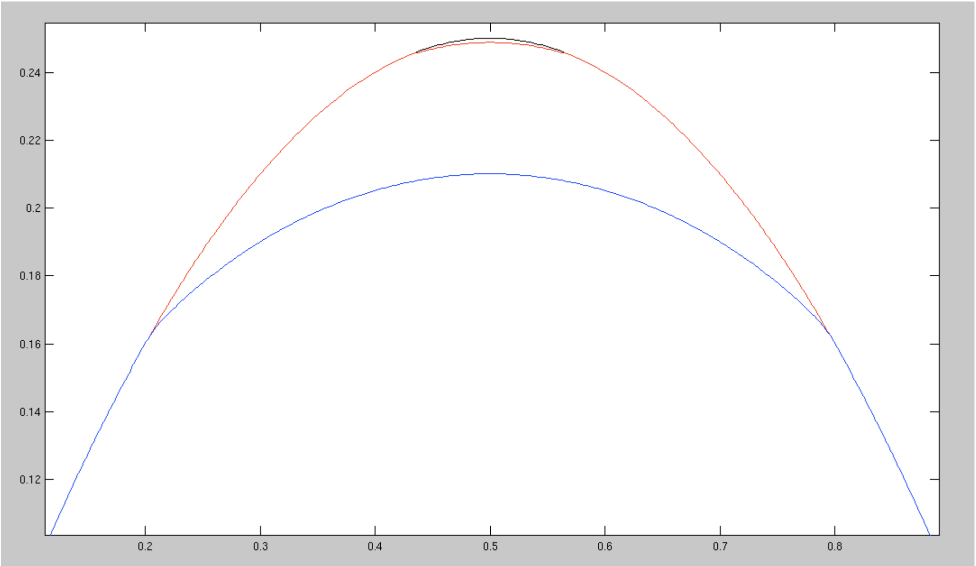
NUMERICS: MAKE ANSATZ $\mathcal{R} = (-r, r)$ FOR SOME $r \geq 0$.

- WITH $\underbrace{\lambda = 2,\ \beta = 0.05}_{\rightsquigarrow\ 40 \text{ MEETINGS / AGENT ON AVE.}}$, $\gamma = 0.05$, $c = 0.0125$,

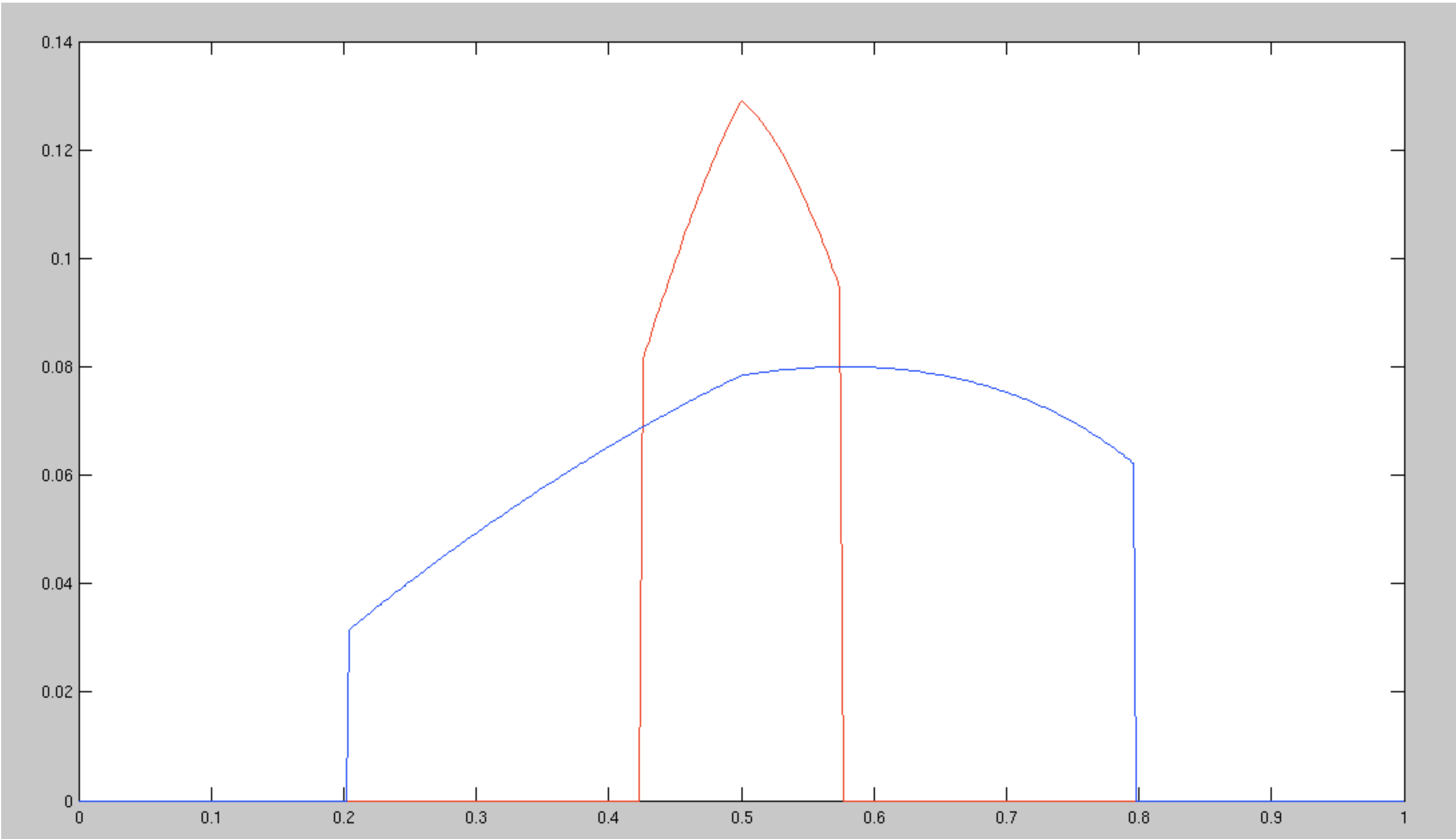
$$\pi^H(d\theta) = \frac{\exp(-(\theta-1)^2/8)\, d\theta}{\int \exp(-(\theta-1)^2/8)\, d\theta},$$

$\mathcal{R} \longmapsto \mathcal{R}^*$ ($r \longmapsto r^*$):





- LARGEST EQUILIBRIUM VALUE IS PARETO-OPTIMAL!



0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

FORWARD SCHEME:

$$\begin{cases} 0 = \lambda\left(\mu * \mu - \mu\left(\int_{-r}^{r}\mu\right)\right) + \beta(\pi - \mu) & \text{IN } (-r,r) \\ \operatorname{supp}\mu \subset (-r,r). \end{cases}$$

⇝ CONTRACTIVE MAP FOR μ WHEN $\beta > 4\lambda$.

- FOR GENERAL $\beta, \lambda > 0$ WRITE AS $\mu = \Phi(\mu)$,

$$\Phi(\mu) = \frac{1}{\beta + \lambda\left(\int_{-r}^{r}\mu\right)}\left(\lambda(\mu * \mu) + \beta\pi\right)\mathbb{1}_{(-r,r)}$$

- $\begin{cases} \mu^{(n+1)} = \Phi(\mu^{(n)}) \\ \mu^{(0)} = \pi\,\mathbb{1}_{(-r,r)} \end{cases}$ CONVERGES FOR ANY $\beta, \lambda > 0$

  (PROOF IN PROGRESS)

OBSTACLE PROBLEM SCHEME:

- RANDOM MATCHING $\rightsquigarrow$ PURE-JUMP PROCESS (COMPOUND POISSON) FOR INDIVIDUAL'S TYPE.

  $\rightsquigarrow$ AGENTS ONLY STOP IMMEDIATELY AFTER JUMP (UNLESS REPLACED BEFORE THEN)

$\Rightarrow$ TIME-DISCRETE PROBLEM!

LET

$$T\varphi(\theta) = \mathbb{E}\left[\int_0^{\tau_1} e^{-(\gamma+\beta)s}\left(c + \beta g(\theta_i(s))\right) ds + e^{-(\gamma+\beta)\tau_1}\varphi(\theta_i(\tau_1))\right]$$

WHERE $\tau_1 \sim \mathrm{Exp}(\lambda)$ IS FIRST JUMP TIME OF $\theta_i(t)$.

$$\max\left\{ -\mathcal{L}v + (\gamma+\beta)v - (c+\beta g),\; v-g \right\} = 0$$

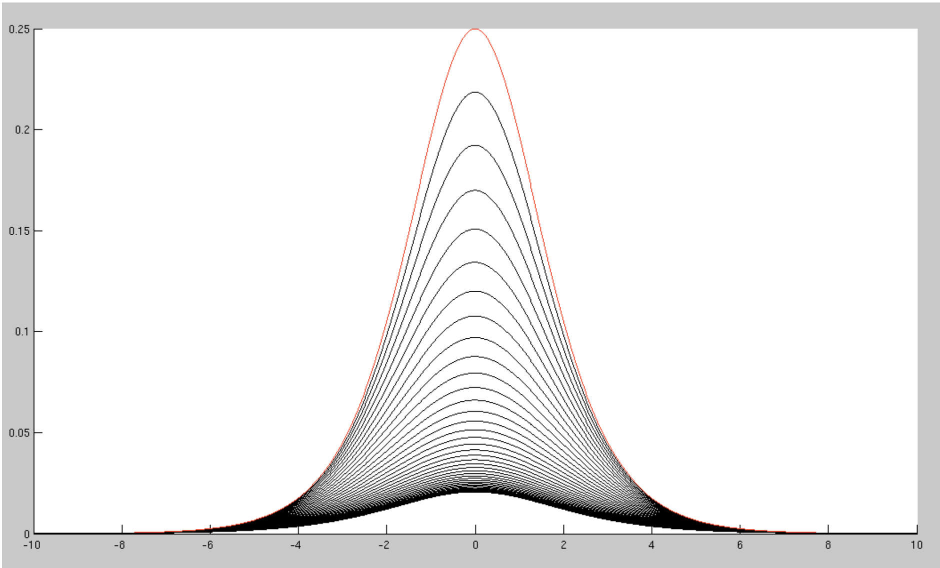
OBSTACLE PROBLEM

$$\Updownarrow$$

$$\min\left\{ Tv,\; g \right\} = v$$

WALD-BELLMAN EQN.

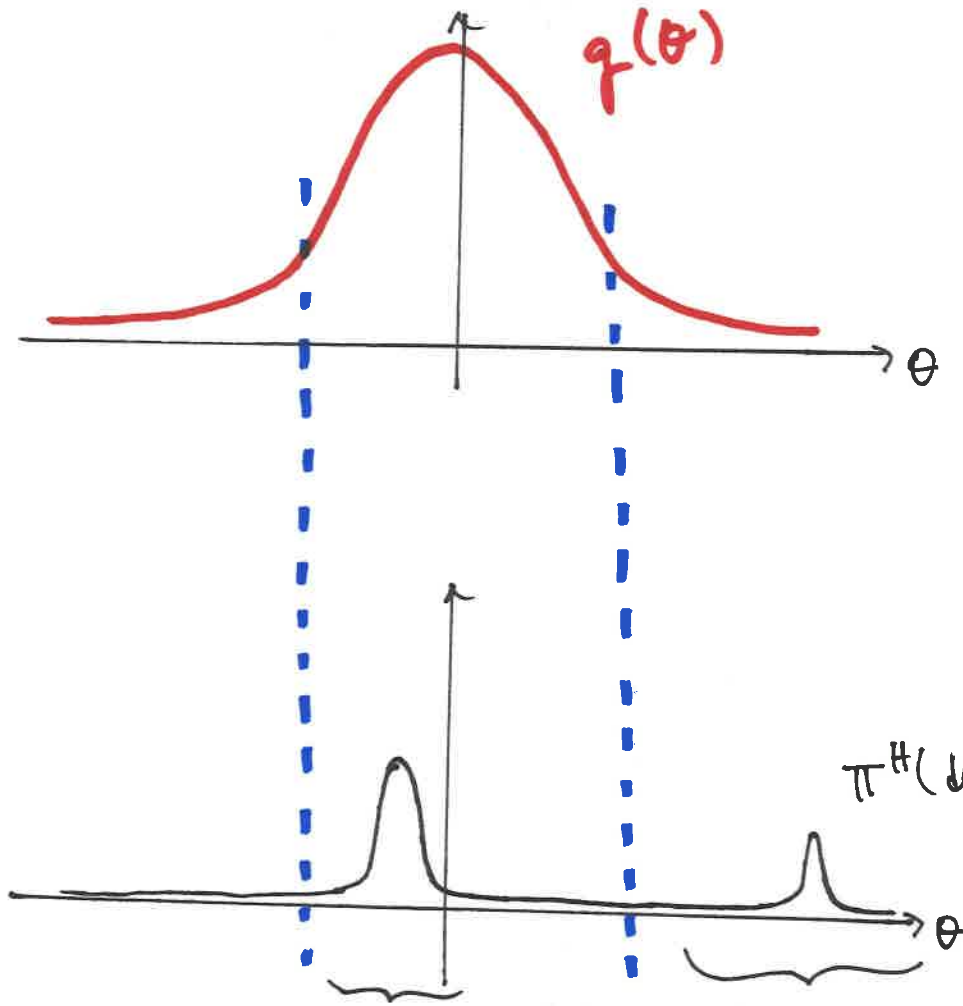
- SCHEME : $\begin{cases} v^{(n+1)} = \min\left\{ Tv^{(n)},\; g \right\} \\ v^{(0)} = g \end{cases} \Longrightarrow v^{(n)} \searrow v$ UNIF.



- SAME PROPERTY THAT LEADS TO KINETIC FORWARD EQN. SIMPLIFIES OPTIMAL STOPPING PROBLEM.

- MODEL ALLOWS FOR RICH CLASS OF EQUILIBRIA

EX. (EDUCATED GET RICHER, POORLY/BADLY EDUCATED GET POORER)



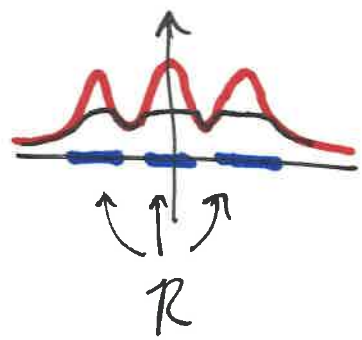
BADLY EDUCATED REMAIN AND LEAVE WITH WRONG INFORMATION.

EDUCATED LEAVE MARKET IMMEDIATELY

WORK IN PROGRESS:

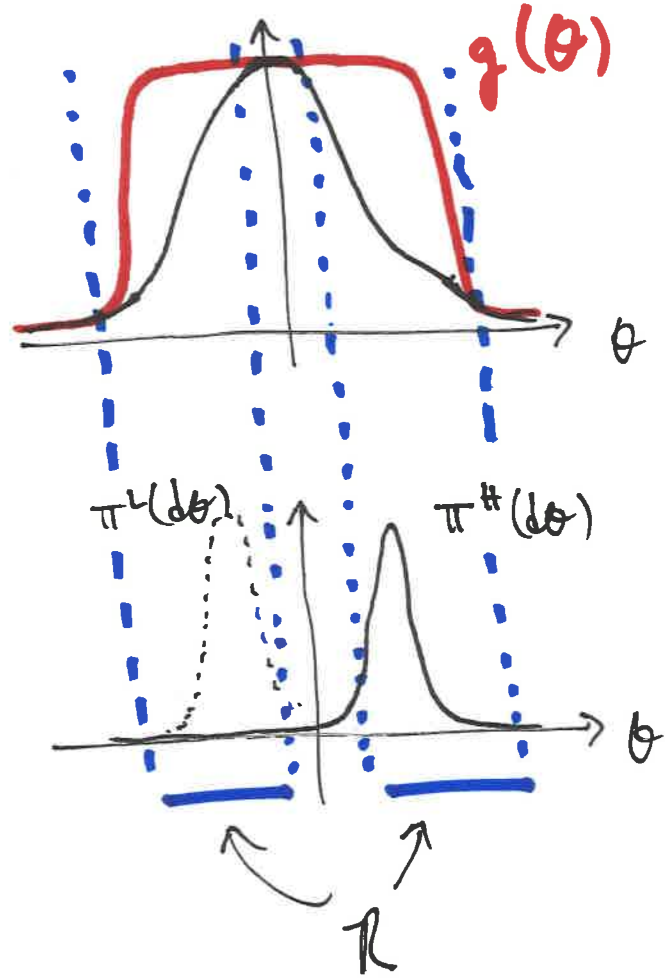
- EXISTENCE THEOREM FOR NASH/MFG EQUILIBRIA.
- HOW "BAD" CAN $\partial\mathcal{R}$ BE?

  $\rightsquigarrow$ AS "BAD" AS OBSTACLE g



- FOR g SIMILAR TO ONE ORIGINALLY CONSIDERED (g'' HAS ONLY ONE CHANGE IN SIGN FOR $\theta > 0$), HYPOTHESIZE THAT $\mathcal{R} = (-r, r) \setminus (-r_{inner}, r_{inner})$ FOR SOME $r, r_{inner} \geq 0$.

INTUITION:



(g'' CHANGES SIGN <u>ONCE</u> FOR $\theta > 0$)

- THOSE AGENTS CLOSE TO SUDDEN DROP IN EXIT COST STAY ACTIVE IN MARKET.
- THOSE COMPLETELY UNINFORMED GIVE UP IMMEDIATELY.

THANKS FOR LISTENING!