Rates of convergence for Smoluchowski's coagulation equation

Ravi Srinivasan
Duke Univ./Univ. of Texas at Austin
Dec. 07, 2009

- R. Srinivasan, http://arxiv.org/abs/0905.3450 (2009)

Smoluchowski's coagulation equation

$n(t,x)$ = number density of clusters of size $x > 0$ at time $t \geq 0$

$K(x,y) = K(y,x)$ symmetric rate kernel

$$\partial_t n(t,x) = \frac{1}{2}\int_0^x K(x-y,y)n(t,x-y)n(t,y)dy - \int_0^\infty K(x,y)n(t,x)n(t,y)dy$$

$$\boxed{x-y} + \boxed{y} \xrightarrow{K(x-y,y)} \boxed{x}$$

$$\boxed{x} + \boxed{y} \xrightarrow{K(x,y)} \boxed{x+y}$$

Smoluchowski's equation been used as a mean-field model for a variety of agglomeration phenomena:

- coagulation of colloids
- formation of clouds and smog
- kinetics of polymerization
- mass aggregation in astrophysics
- schooling of fishes
- merging of banks
- random graph theory
- ballistic aggregation of shocks in Burgers turbulence ($K=x+y$)

More generally, consider measure-valued solutions under weak formulation (moment identity), with suitable test functions ϕ:

$n(t, dx)$ = number measure of clusters of size in $[x, x + dx)$

$$\partial_t \int_{(0,\infty)} \phi(x) n(t, dx)$$
$$= \frac{1}{2} \int_{(0,\infty)} \int_{(0,\infty)} (\phi(x+y) - \phi(y) - \phi(x)) K(x,y) n(t,dy) n(t,dx)$$

Many recent studies have focused on the homogeneous, "solvable" rate kernels

$$K(\alpha x, \alpha y) = \alpha^\gamma K(x,y), \forall \alpha > 0$$

$$K = 2, x+y, xy$$

$$\gamma = 0, 1, 2$$

• For these kernels one can obtain exact solutions via the Laplace transform

dynamic scaling

↙↗ ↖↘

Smoluchowski's coagulation equation	classical limit theorems in probability
$n(t, dx)$	$S_n = X_1 + ... + X_n$

(i) Well-posedness of dynamical system on space of prob. measures

- Menon & Pego (CPAM, 2004)
- Well-posedness for general homogeneous K:

Fournier & Laurençot, Escobedo; Mischler & Rodriguez Ricard

(ii) Existence of 1-parameter family of self-similar solutions, domains of attraction:

- Menon & Pego (CPAM, 2004)

SSS w/exp. decay	⟷	Gaussian
SSS w/power law decay	⟷	Lévy stable laws
convergence to SSS, regular variation	⟷	central limit theorem

dynamic scaling

Smoluchowski's coagulation equation

$n(t, dx)$

classical limit theorems in probability

$S_n = X_1 + ... + X_n$

(iv) Interplay between moment hypothesis on the initial data and stronger modes of convergence

• Menon & Pego (SIAM Review, 2006): For SSS with exp. decay, uniform convergence of densities under dynamic scaling

There are more correspondences, which we do not discuss here:

(v) Attractor of the dynamical system modulo scaling

• Menon & Pego (2008)

eternal solutions ⟷ infinitely divisible distributions

Continuing the analogy...

rates of convergence to SSS with exponential tail

$\longleftrightarrow$

Berry-Esséen theorem for convergence to Gaussian

$$\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = 1, \mathbb{E}X_i^3 = \rho < \infty$$

$$F_n(x) = \mathbb{P}\left(\frac{X_1 + \ldots + X_n}{\sqrt{n}} \leq x\right)$$

$$\Downarrow$$

$$\sup_{x\in\mathbb{R}} |F_n(x) - \mathcal{N}(x)| \leq \frac{3\rho}{\sqrt{n}}$$

Continuing the analogy...

rates of convergence to SSS with exponential tail

?

↔

Berry-Esséen theorem for convergence to Gaussian

$$\mathbb{E}X_i = 0, \mathbb{E}X_i^2 = 1, \mathbb{E}X_i^3 = \rho < \infty$$

$$F_n(x) = \mathbb{P}\left(\frac{X_1 + ... + X_n}{\sqrt{n}} \leq x\right)$$

⬇

$$\sup_{x \in \mathbb{R}} |F_n(x) - \mathcal{N}(x)| \leq \frac{3\rho}{\sqrt{n}}$$

- Initial time $t_0 = 1 \quad (\gamma = 0)$
  $t_0 = 0 \quad (\gamma = 1, 2)$

$$\int_{(0,\infty)} x^{\gamma} n(t_0, dx) = 1 < \infty \longrightarrow \text{well-posedness}$$

- Dynamic scaling

$$m_\gamma(t) = \int_{(0,\infty)} x^\gamma n(t, dx)$$

$$m_0(t) = t^{-1}, \qquad m_1(t) = 1, \qquad m_2(t) = (1-t)^{-1}$$

$$\lambda_0(t) = t, \qquad \lambda_1(t) = e^{2t}, \qquad \lambda_2(t) = (1-t)^{-2}$$

- Self-similar solution with exponential tail

$$n(t, dx) = \frac{m_\gamma(t)}{\lambda_\gamma(t)^\gamma} \hat{n}_{*,\gamma}(d\hat{x}) \qquad \hat{x} = \frac{x}{\lambda_\gamma(t)}$$

$$\hat{n}_{*,0}(d\hat{x}) = e^{-\hat{x}} d\hat{x} \qquad \hat{x}\hat{n}_{*,1}(\hat{x}) = \hat{x}^2\, \hat{n}_{*,2}(\hat{x}) = \frac{1}{\sqrt{2\pi}} \hat{x}^{-1/2} e^{-\hat{x}/2} d\hat{x}$$

$$\int_{(0,\infty)} x^{\gamma+1} n(t_0, dx) = 1 < \infty \longrightarrow$$ in domain of attraction of SSS with exponential tail

- Time change

$$\tau_\gamma(t) = \int_{t_0}^{t} m_\gamma(s) ds$$

$$\tau_0(t) = \log(t), \qquad \tau_1(t) = t, \qquad \tau_2(t) = \log(1-t)^{-1}$$

- Rescaled solution converges to SSS (weak convergence of measures)

$$\hat{n}(\tau_\gamma, d\hat{x}) := \frac{\lambda_\gamma(t)^\gamma}{m_\gamma(t)} n(t, \lambda_\gamma(t) d\hat{x})$$

$$\hat{n}(\tau_\gamma, d\hat{x}) \xrightarrow{\tau_\gamma \to \infty} \hat{n}_{*,\gamma}(d\hat{x})$$

$$\int_{(0,\infty)} x^{\gamma+2} n(t_0, dx) := \mu_{\gamma+2} < \infty \longrightarrow$$ convergence rate

- Exponential convergence in terms of distribution functions

$$F_\gamma(\tau_\gamma, \hat{x}) = \int_{(0,\hat{x})} \hat{y}^\gamma \hat{n}(\tau_\gamma, d\hat{y}), \qquad F_{*,\gamma}(\hat{x}) = \int_{(0,\hat{x})} \hat{y}^\gamma \hat{n}_{*,\gamma}(d\hat{y})$$

Theorem (Srinivasan, 2009): For any $\tau_\gamma(t) \in [0, \infty)$

$$\sup_{\hat{x}>0} |F_\gamma(\tau_\gamma, \hat{x}) - F_{*,\gamma}(\hat{x})| \leq C(\mu_{\gamma+2})(1+\tau_\gamma)e^{-\tau_\gamma}$$

• Holds for a broad class of initial data with minimal assumptions (existence of an additional higher moment)

• Near optimal: For monodisperse initial data $n_0(dx) = \delta_1(dx)$,

$$\sup_{\hat{x}>0} |F_\gamma(\tau_\gamma, \hat{x}) - F_{*,\gamma}(\hat{x})| = O(e^{-\tau_\gamma})$$

• For K=2: Cañizo, Mischer, Mouhot (2008) showed exponential convergence to SSS in a weighted Sobolev norm for initial densities satisfying decay assumptions on its derivatives, also using Fourier methods

Main ingredients of proof for K=2:

- Consider Fourier-Laplace representation of number measure

$$s \in \bar{\mathbb{C}}_+ := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$$

$$u(\tau, s) = \int_{(0,\infty)} e^{-s\hat{x}} \hat{n}(\tau, d\hat{x}), \qquad u_*(s) = \int_{(0,\infty)} e^{-s\hat{x}} \hat{n}_*(d\hat{x})$$

- Smoothing argument (Feller, Vol. II)

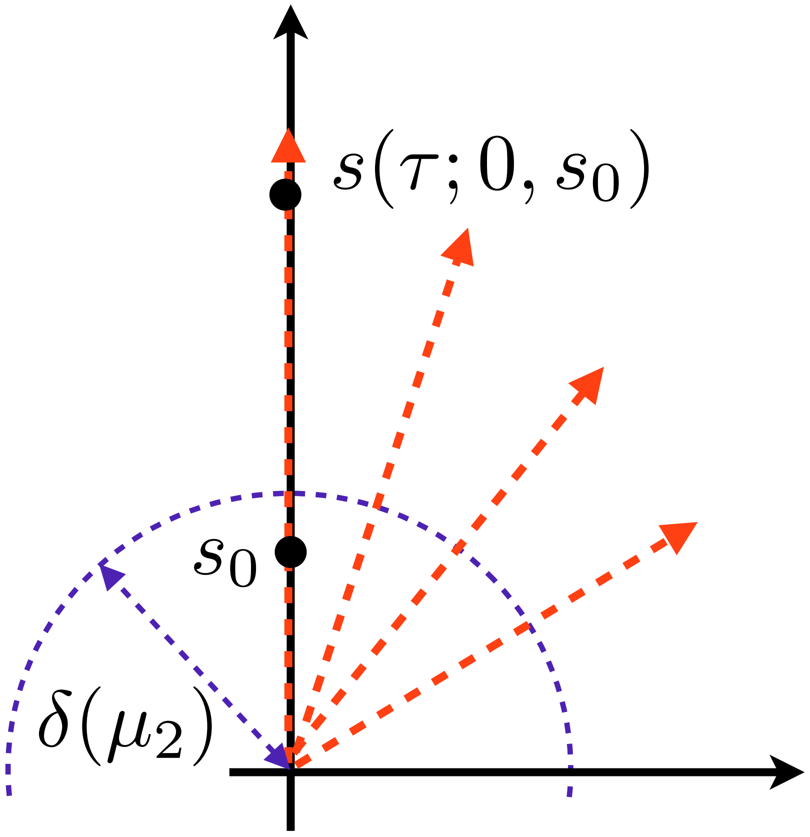
$$\sup_{\hat{x}>0} |F(\tau, \hat{x}) - F_*(\hat{x})| \leq \frac{1}{\pi} \sup_{\hat{x}>0} \left| \int_{-iT}^{iT} \frac{e^{s\hat{x}}}{s} (u(\tau, s) - u_*(s)) ds \right| + \frac{24}{\pi T}$$

↑ main term to estimate

(i) Smoluchowski's equation ‹—› PDE for Laplace transform

$$\partial_\tau u + s\partial_s u = -u(1-u)$$

For K=2, characteristics $s(\tau;0,s_0) = e^\tau s_0$ *do not depend on initial data.*



$$u(\tau,s) = e^{-\tau}\frac{u_0(s_0)}{1-u_0(s_0)(1-e^{-\tau})}$$

$$u_*(s) = e^{-\tau}\frac{u_*(s_0)}{1-u_*(s_0)(1-e^{-\tau})}$$

Main term with $T = \delta e^\tau$:

$$\left|\int_{-i\delta}^{i\delta}\frac{e^{s(\tau;0,s_0)\hat{x}}}{s(\tau;0,s_0)}\frac{(u_0(s_0)-u_*(s_0))}{(1-u_*(s_0)(1-e^{-\tau}))(1-u_0(s_0)(1-e^{-\tau}))}ds_0\right|$$

(ii) Moment hypothesis (decay of tails of initial data) gives approximation for difference of Laplace transforms near origin

$$\begin{aligned} u_0(0) &= -u_0'(0) = 1, \qquad & u_0''(0) &= \mu_2 \\ u_*(0) &= -u_*'(0) = 1, \qquad & u_*''(0) &= 2 \end{aligned}$$

$$|u_0(s_0) - u_*(s_0)| \le \left(1 + \frac{\mu_2}{2}\right) |s_0|^2$$

This approximation is good in the region $|s_0| \le \delta(\mu_2)$ with

$$2\delta(\mu_2) = \sqrt{1 + 2\left(1 + \frac{\mu_2}{2}\right)^{-1}} - 1$$

(iii) Plug in estimates—main contribution:

$$\begin{aligned} \left| \int_{\delta e^{-\tau}}^{\delta} \right| &\le 2\left(1 + \frac{\mu_2}{2}\right)(1+\delta)^2 e^{-\tau} \int_{\delta e^{-\tau}}^{\delta} \frac{1}{|s_0|} d|s_0| \\ &= 2\left(1 + \frac{\mu_2}{2}\right)(1+\delta)^2 \tau e^{-\tau} \end{aligned}$$

done!

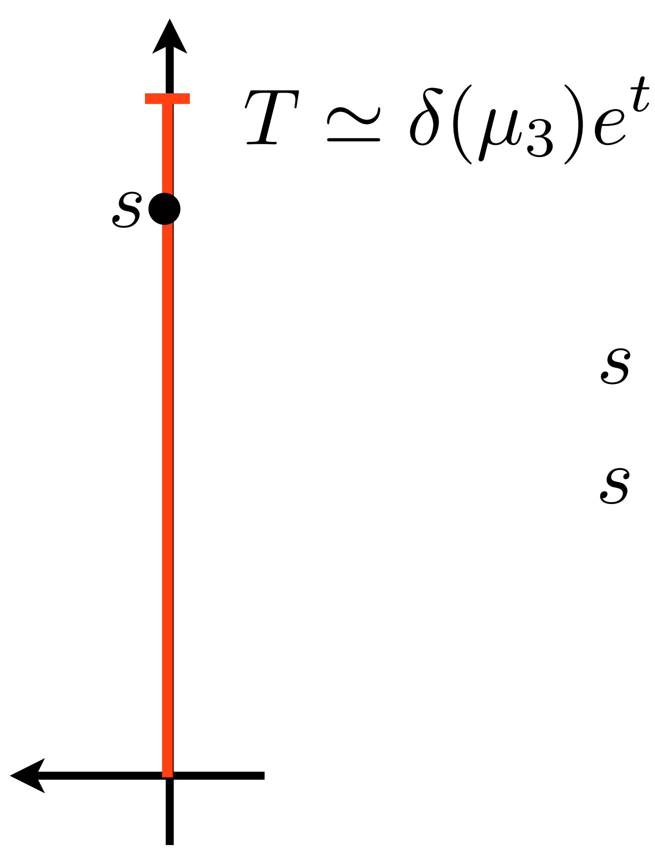
What about K=x+y?

• Idea of proof is same as for K=2. But characteristics depend on initial data and are no longer rays, but curves in the complex plane. We use a contour deformation argument as in Menon-Pego (2006):

$$\varphi(t,s) = \int_{(0,\infty)} (1 - e^{-s\hat{x}})\hat{n}(t, d\hat{x})$$
$$u(t,s) = \int_{(0,\infty)} e^{-s\hat{x}}\hat{x}\hat{n}(t, d\hat{x})$$

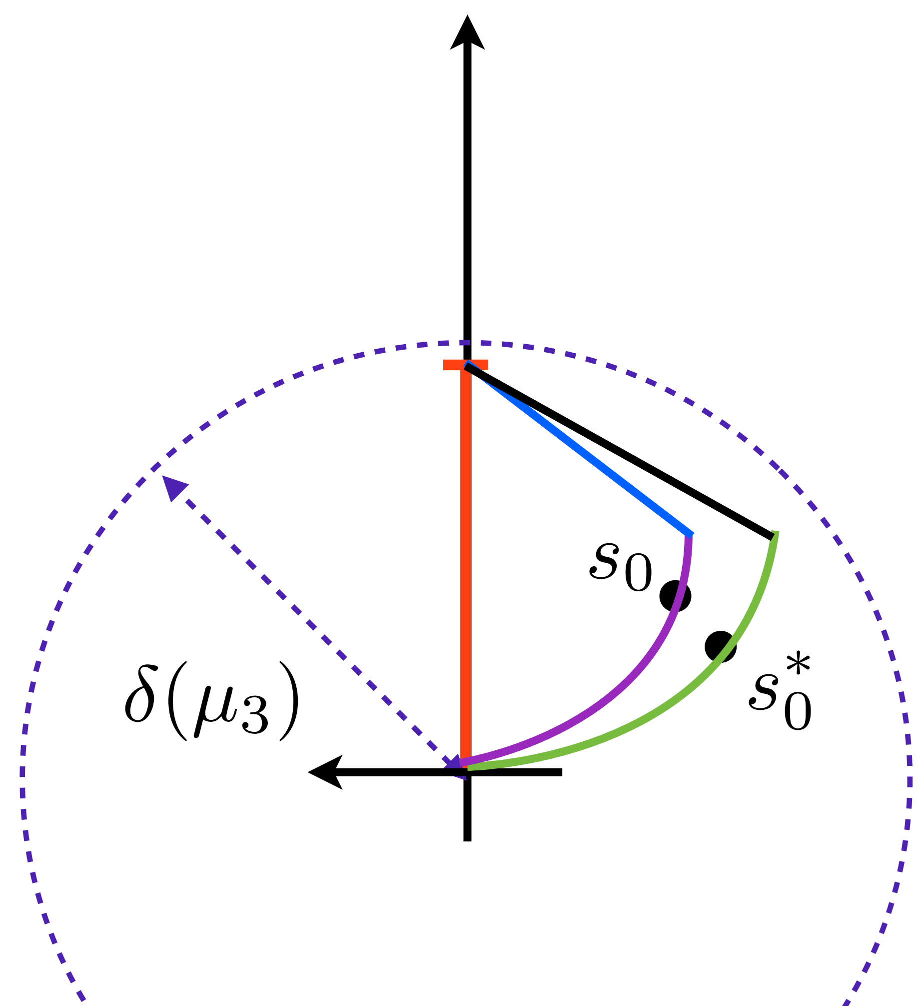
$\longrightarrow$

$$\partial_t \varphi + (2s - \varphi)\partial_s \varphi = \varphi$$
$$\partial_t u + (2s - \varphi)\partial_s u = -u(1-u)$$



$s \longrightarrow s_0$

$s \longrightarrow s_0^*$



What about K=xy?

• Solutions can be obtained from the case K=x+y by a well-known change of variables given in Drake (1972). We therefore get the convergence rate for K=xy for free.

dynamic scaling

↕

Smoluchowski's coagulation equation

$n(t, dx)$

classical limit theorems in probability

$S_n = X_1 + ... + X_n$

Many thanks to Govind Menon (Brown) & Bob Pego (CMU)